Mathematical Analysis

Chapter 12
Sequences of Functions in Metric Spaces

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Overview

In this chapter, we study properties of sequences of functions in general metric spaces. We will only concern ourselves with number sequences when their study advances our study of sequences of functions.

Notation: The symbol $\mathbb{K}$ is sometimes used to denote either $\mathbb{R}$ or $\mathbb{C}$.

$C_{\mathbb{R}}([0,1])$ is then $\mathbb{R}-$vector space of continuous functions $[0,1] \mapsto \mathbb{R}$.

$\mathcal{F}_{\mathbb{R}}([0,1])$ is then $\mathbb{R}-$vector space of functions $[0,1] \mapsto \mathbb{R}$.

$\mathcal{R}_{\mathbb{R}}([0,1])$ is then $\mathbb{R}-$vector space of Riemann-int. functions $[0,1] \mapsto \mathbb{R}$.

$\mathcal{C}_c(\mathbb{R},\mathbb{C})$ is the set of continuous functions with compact support.

Outline

12.1 – Uniform Convergence

Let X be a set and let (E, d) be a metric space. A sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n : X \to E$ is said to **converge pointwise** to a function $f : X \to E$ (denoted by $f_n \to f$ on X) if $f_n(\mathbf{x}) \to f(\mathbf{x})$ for all $\mathbf{x} \in X$.

Symbolically, $f_n \to f$ on X if

$$\forall \varepsilon > 0, \forall \mathbf{x} \in X, \exists N = N_{\varepsilon,\mathbf{x}} \text{ such that } n > N \implies d(f_n(\mathbf{x}), f(\mathbf{x})) < \varepsilon$$

(note the **explicit dependence** of N on $\mathbf{x}$).

As we have discussed in chapters 6 and 7, pointwise convergence is quite often not strong enough.

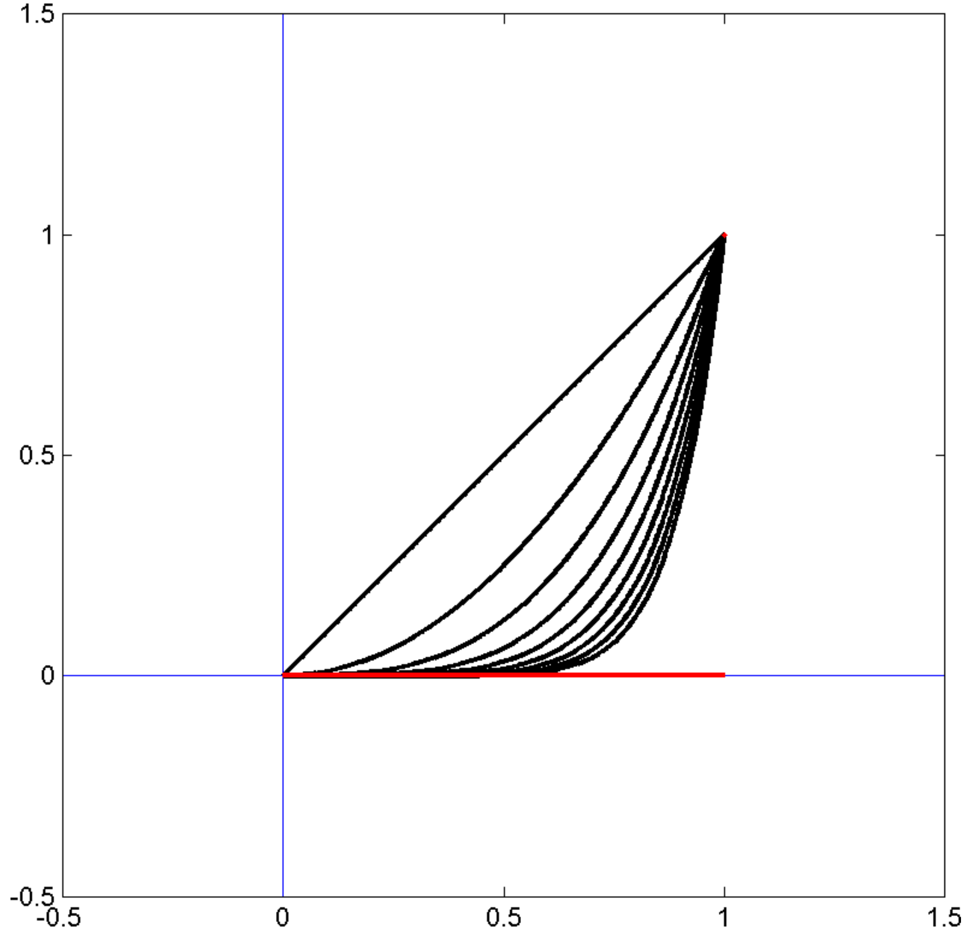
Consequently, we introduce a second kind of convergence: the sequence (f_n) is said to **converge uniformly** to a function $f : X \to E$ (denoted by $f_n \rightrightarrows f$ on X) if we can remove the explicit dependence of N on $\mathbf{x}$.

Symbolically, $f_n \rightrightarrows f$ on X if

$$\forall \varepsilon > 0, \exists N = N_\varepsilon \text{ such that } n > N \implies \sup_{\mathbf{x}\in X}\{d(f_n(\mathbf{x}), f(\mathbf{x}))\} < \varepsilon.$$

Examples:

1. Let $(E,d) = (\mathbb{R}, |\cdot|)$, $X = [0,1]$ and $f_n : X \to E$ be defined by $f_n(x) = x^n$.



The sequence (f_n) in black, the limit f in red.

2. With the definitions as in the last example,

Proposition 139. (CAUCHY'S CRITERION)
Let (E, d) be a complete metric space and let (f_n) be a sequence of functions $f_n : X \to E$. Then, $f_n \rightrightarrows f$ on X if and only if

$$\forall \varepsilon > 0, \exists N = N_\varepsilon > 0 \text{ s.t. } n, m > N \implies \sup_{\mathbf{x} \in X} \{d(f_n(\mathbf{x}), f_m(\mathbf{x}))\} < \varepsilon.$$

Proof.

■

In order to lighten the text, we will sometimes write $\|d(f_n, f_m)\|_\infty$ for

$$\sup_{\mathbf{x}\in X}\{d(f_n(\mathbf{x}), f_m(\mathbf{x}))\}.$$

Similar notions exist for **series**. Let (E, d) be a metric space and let (u_n) be a sequence of functions $u_n : X \to E$.

For any $m \in \mathbb{N}$, define the **partial sum** $f_m : X \to E$ by

$$f_m(\mathbf{x}) = u_1(\mathbf{x}) + \cdots + u_m(\mathbf{x}) = \sum_{n=1}^{m} u_n(\mathbf{x}).$$

The sequence (f_m) is the **series generated** by (u_n), and it is usually denoted by $\displaystyle\sum_{n\in\mathbb{N}} u_n$.

If $f_m \to f$ on X, we say that the series **converges (pointwise)** on X.

If $f_m \rightrightarrows f$ on X, we say that the series **converges uniformly** on X.

In both cases, f is said to be the **sum** of the series.

If (f_m) does not converge, we say that the series **diverges**.

Let E be a Banach space and let (g_n) be a sequence of functions $g_n \in B(X, E)$. The series $\sum g_n$ **converges absolutely** on X if $\sum \|g_n\|_\infty$ converges (note that there is no need to stipulate the type of convergence in the latter case).

Proposition 140. *If $\sum g_n$ converges absolutely on X, then $\sum g_n$ converges uniformly on X.*

Proof.

∎

12.1.1 – Properties

The two main types of convergence are not created equal, however. We establish the superiority of uniform convergence over pointwise convergence in a series of well-known theorems.

Theorem 141. *Let* (E, d) *and* $(F, \tilde{d})$ *be metric spaces. If* $(f_n) \subseteq \mathcal{C}(E, F)$ *is such that* $f_n \rightrightarrows f$ *on* E*, then* $f \in \mathcal{C}(E, F)$*.*

Proof.

■

We have already seen an example showing that this does not necessarily hold for pointwise convergence.

Theorem 142. (Limit Interchange; R-Integrable Functions) *Let $(E, \|\cdot\|)$ be a Banach space. If $(f_n) \subseteq \mathcal{F}([a,b], E)$ is such that $f_n \rightrightarrows f$ on $[a,b]$, and if f_n is Riemann-integrable over $[a,b]$ for all n, then f is Riemann-integrable and $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$.*

Proof. ■

The fact that the limit interchange is not necessarily valid if $f_n \to f$ instead of $f_n \rightrightarrows f$ on $[a,b]$ could be seen as an indictment of the Riemann-integral rather than as an indictment of pointwise convergence. In a coming chapter, we will take the former position and introduce the **Lebesgue integral** to circumvent this difficulty.

The next result is a companion to Theorem 142.

Theorem 143. (Limit Interchange; Fundamental Theorem) *Let $(E, \|\cdot\|)$ be a Banach space. If $(f_n) \subseteq \mathcal{F}([a,b], E)$ is such that $f_n \rightrightarrows f$ on $[a,b]$, and if f_n is Riemann-integrable over $[a,b]$ for all n, then f is Riemann-integrable according to Theorem 142.*

Define $F_n, F : [a,b] \to E$ by $F_n(x) = \int_a^x f_n(t)\,dt$ and $F(x) = \int_a^x f(t)\,dt$. Then $F_n \rightrightarrows F$ on $[a,b]$.

Proof.

■

Theorem 143 has an interesting corollary when applied to series, which is often assumed to hold (without proof) when solving differential equations.

Theorem 144. *Let $(E, \|\cdot\|)$ be a Banach space and let $\sum g_n$ be a series of functions in $\mathcal{R}([a,b],E)$. If $\sum g_n$ is uniformly convergent, then*

$$\int_a^b \left(\sum_{n\in\mathbb{N}} g_n(t) \right) dt = \sum_{n\in\mathbb{N}} \left(\int_a^b g_n(t)\,dt \right).$$

Proof. ■

Theorem 145. (Limit Interchange; Differentiable Functions) *Let $(E, \|\cdot\|)$ be a Banach space. If $(f_n) \subseteq \mathcal{C}^1([a,b],E)$ is such that $f_n(x_0) \to f(x_0)$ for some $x_0 \in [a,b]$ and if $\exists g \in \mathcal{C}([a,b],E)$ such that $f_n' \rightrightarrows g$ on $[a,b]$, then $\exists f \in \mathcal{C}^1([a,b],E)$ such that $f_n \rightrightarrows f$ on $[a,b]$ and $f' = g$.*

Proof.

∎

Examples:

1. Compute $\int_0^\infty f(x)\,dx$, where $f(x) = \frac{x^2}{\exp(x)-1}$.

   Solution.

2. Show that uniform convergence is not equivalent to absolute convergence.

   Proof.

   ∎

12.1.2 – Abel's Criterion

A number of tests can be used to gauge the convergence of series (whether numerical series or series of functions).

From calculus, you may remember the following tests:

- $p-$test;
- comparison test;
- alternating series test;
- integral test;
- d'Alembert test (also known as the ratio test), or
- Cauchy test (also known as the root test).

In this section, we present a new test for convergence of a series.

Proposition 146. (ABEL'S CRITERION)
Let $(\mathbf{a}_n) \subseteq E$, where E is a Banach space over $\mathbb{R}$. Suppose that we can write $\mathbf{a}_n = \varepsilon_n \mathbf{b}_n$ with

1. *$\varepsilon_n \searrow 0$ a sequence in $\mathbb{R}$, and*
2. *$\exists \sigma \in \mathbb{R}$ such that $\|\sum_{n \leq N} \mathbf{b}_n\| \leq \sigma$ for all $N \in \mathbb{N}$.*

Then $\sum \mathbf{a}_n$ is pointwise convergent and $\|\sum_{n \geq N} \mathbf{a}_n\| \leq 2\sigma\varepsilon_N$ for all $N \in \mathbb{N}$.

Proof.

■

We can easily generalize this result to sequences of functions.

Proposition 147. (ABEL'S CRITERION (REPRISE))
Let $\sum f_n$ be a series of functions $f_n = \varepsilon_n g_n \in \mathcal{F}([a, b], E)$, where E is a Banach space over $\mathbb{R}$. If

1. $\varepsilon_n(x) \searrow 0$ *for all* $x \in [a, b]$;
2. $\exists \sigma \in \mathbb{R}$ *such that* $\| \sum_{n \leq N} g_n(x) \| \leq \sigma$ *for all* $N \in \mathbb{N}$ *and all* $x \in [a, b]$, *and*
3. $\|\varepsilon_n\|_\infty \to 0$.

Then $\sum f_n$ is uniformly convergent on $[a, b]$.

Proof. ■

The three conditions are actually independent (see Exercise 7).

Example: Consider the series $\sum_{k\in\mathbb{N}} c_k b_k(x)$, where $b_k(x) = e^{ikx}$, $x \in \mathbb{R}$ and $c_k \searrow 0$. Show that the series converges (pointwise) for any $x \in (0, 2\pi)$ and that it converges uniformly on $[\delta, 2\pi - \delta]$ for any $\delta \in (0, \pi)$.

Proof.

■

12.2 – Fourier Series

The series $\sum_{k\in\mathbb{N}} c_k e^{ikx}$ in the previous example is continuous on $(0,2\pi)$ even though it fails to converge uniformly on $(0,2\pi)$.

It is an example of a **Fourier Series**, a monumental idea in the development of modern mathematics. They were first proposed as solutions to the **Heat Equation**, a partial differential equation.

In a nutshell, these infinite series gave rise to finite already-known solutions of the Heat Equation, leading the process with which they were formed to be accepted rather hastily as valid, even though a number of mathematicians had an awful lot of objections concerning the use of infinity and (possibly divergent) series (these notions were not as clearly understood back then).

The importance of rigour in mathematics was just starting to be understood by some of the best mathematical minds; while these objections may sound a bit odd nowadays, it is important to remember that the current definitions of the concepts that made some of our predecessors queasy have been distilled of all offending material after years of polishing, which was driven by the very objections that they brought up.

It is no exaggeration to say that Analysis would not be what it is today without this particular episode; while it remains in fashion amongst some mathematicians to deride engineers and physicists for "playing with tools beyond their understanding", let us keep in mind that analytical advances mostly arise from the application of mathematics to so-called 'real-world' problems, in the grand tradition of Archimedes and Newton.

In this section, we introduce and discuss the convergence of Fourier Series.

12.2.1 – Trigonometric Series and Periodic Functions

A **trigonometric polynomial** is any (finite) linear combination of positive powers of sines and cosines:

$$p(t) = a_0 + \sum_{k=1}^{n} \left(a_k \cos(kt) + b_k \sin(kt)\right), \quad \text{where } a_k, b_k \in \mathbb{C}.$$

Since

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i},$$

we can write

$$p(t) = a_0 + \sum_{k=1}^{n} \left(a_k \cos(kt) + b_k \sin(kt)\right) = \sum_{k=-n}^{n} c_k e^{ikt},$$

with

$$a_0 = c_0, \quad a_k = c_k + c_{-k}, \quad \text{and} \quad b_k = i(c_k - c_{-k}),$$

or

$$c_0 = a_0, \quad c_k = \frac{a_k - ib_k}{2}, \quad \text{and} \quad c_{-k} = \frac{a_k + ib_k}{2},$$

for all $1 \leq k \leq n$.

A **trigonometric series** is a formal expression of the form

$$\sum_{k \in \mathbb{Z}} c_k e^{ikt} = a_0 + \sum_{k \in \mathbb{N}} \left(a_k \cos(kt) + b_k \sin(kt) \right).$$

We say that a series indexed by $\mathbb{Z}$ **converges** if both the series indexed by the non-negative integers AND the series indexed by the negative integers converges.

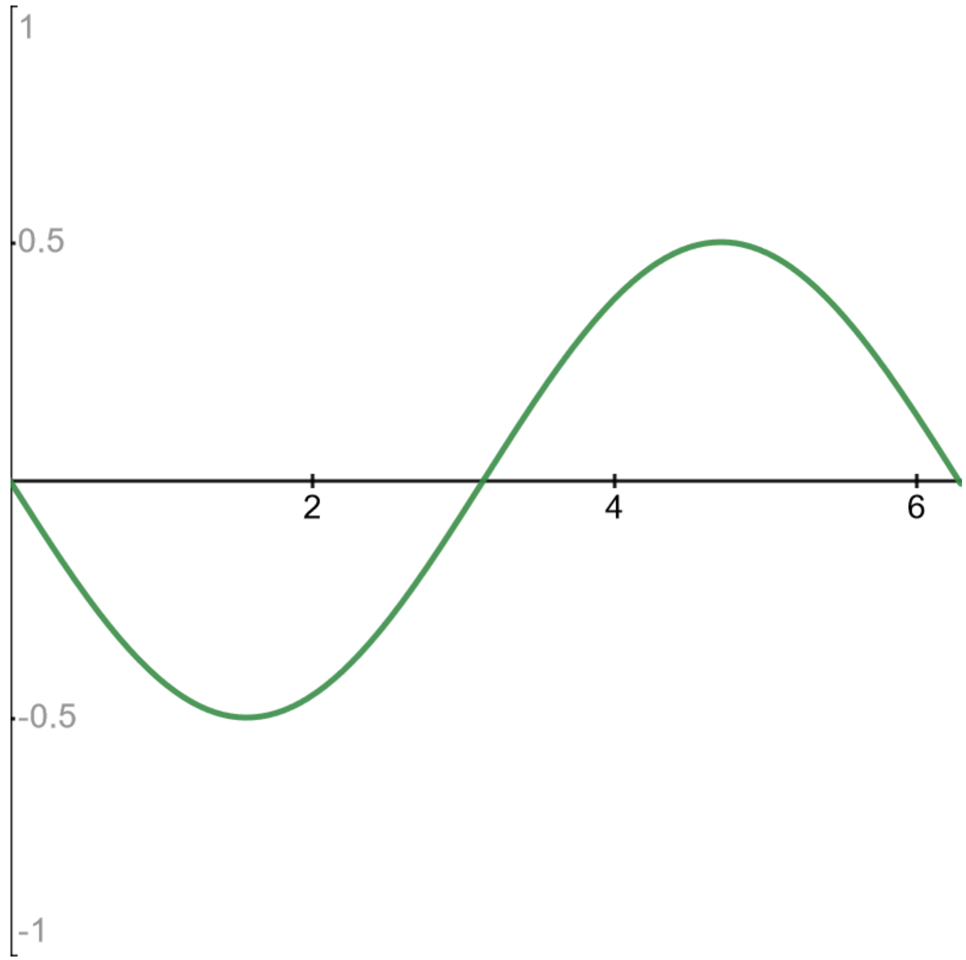
Proposition 148. *If $\sum_{k\in\mathbb{Z}} c_k e^{ikt}$ converges absolutely for some t, then $\sum_{k\in\mathbb{Z}} |c_k| < \infty$. Furthermore, if $\sum_{k\in\mathbb{Z}} |c_k| < \infty$, then $\exists f \in \mathcal{C}(\mathbb{R},\mathbb{C})$ such that $\sum_{k\in\mathbb{Z}} c_k e^{ikt} \rightrightarrows f$ on $\mathbb{R}$.*

Proof. ■

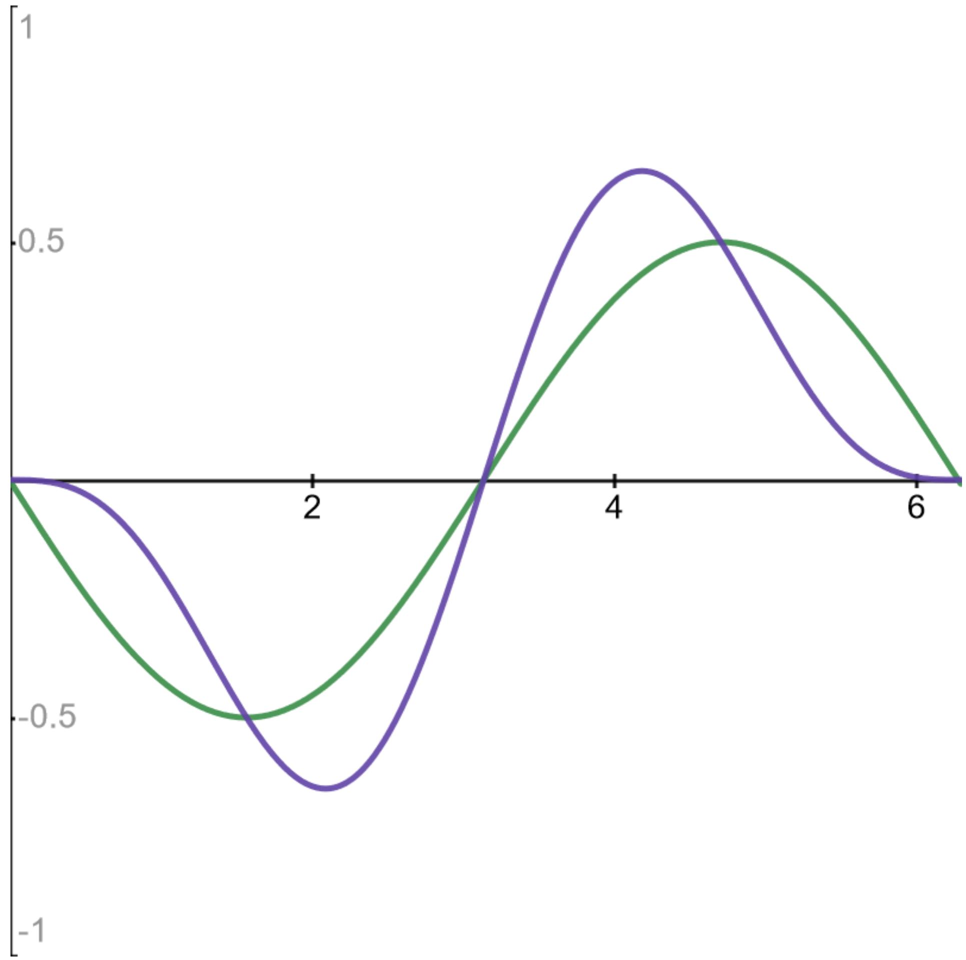
Example: Let $b \in (-1,1)$. Consider the trigonometric series $\sum_{k=1}^{\infty} b^k \sin(kt)$. What is its complex form? Does it converge anywhere? If so, what to?

Solution.

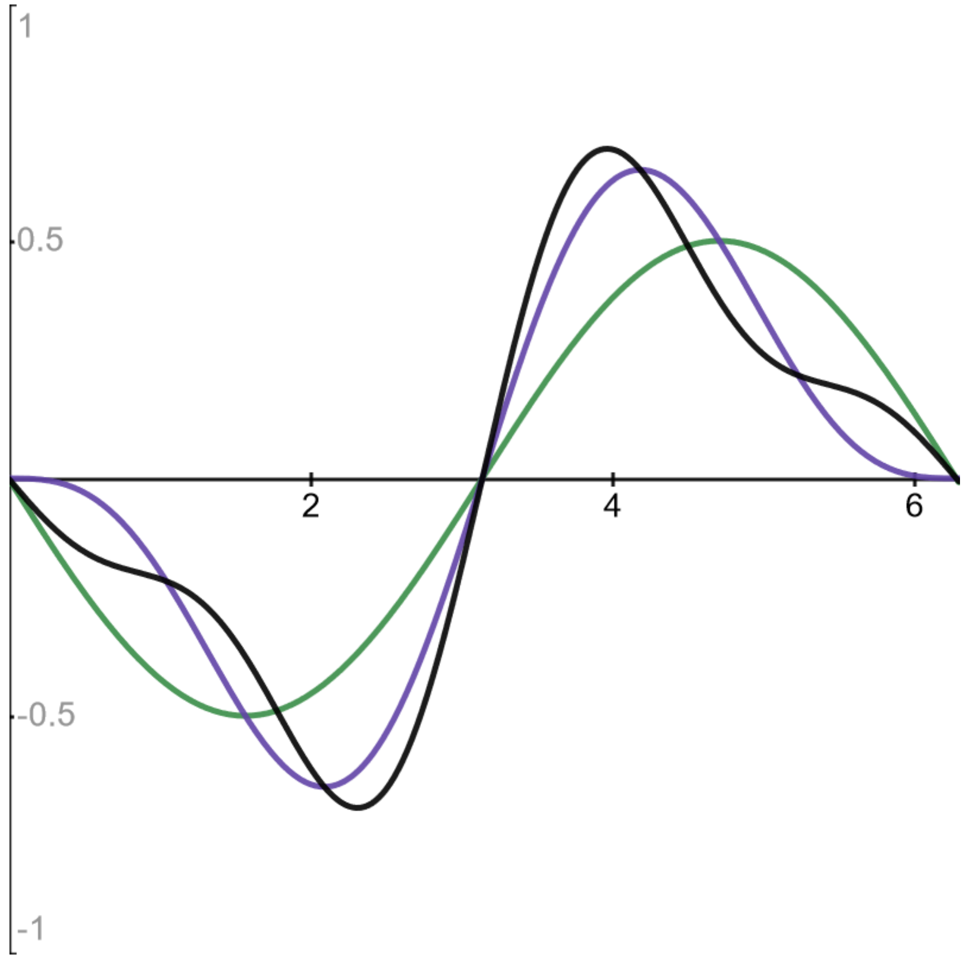
■



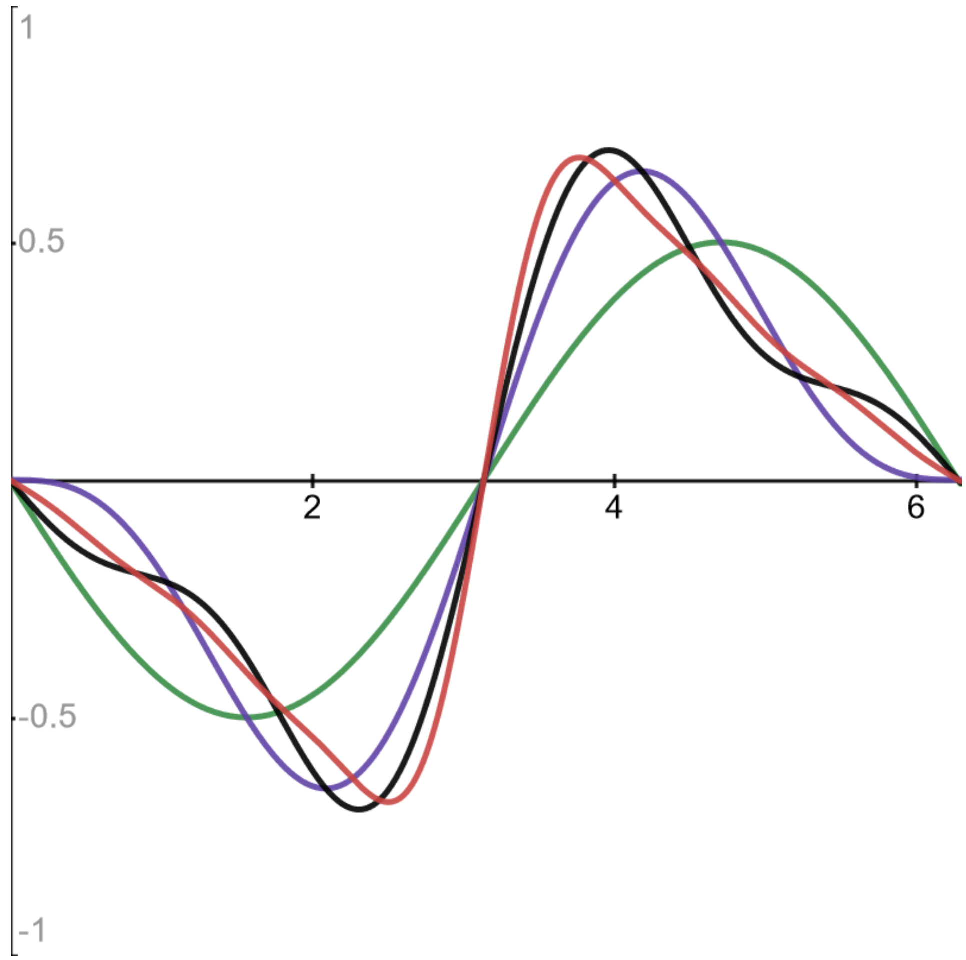
$$-\frac{1}{2}\sin t$$



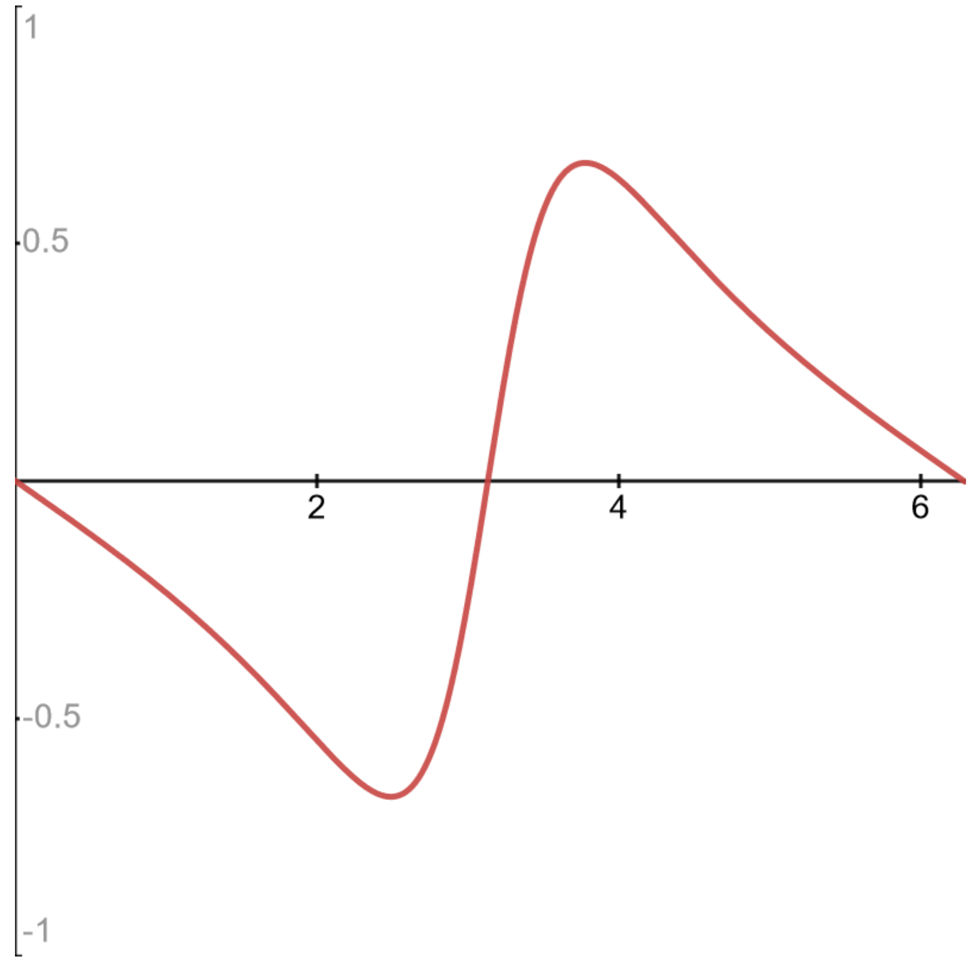
$$-\frac{1}{2}\sin t + (-\tfrac{1}{2})^2 \sin(2t)$$



$$-\tfrac{1}{2}\sin t + (-\tfrac{1}{2})^2 \sin(2t) + (-\tfrac{1}{2})^3 \sin(3t)$$



$$-\frac{\sin t}{2(1+\cos t+(1/2)^4)}$$



$$-\frac{\sin t}{2(1+\cos t+(1/2)^4)}$$

12.2.2 – Again, Abel's Criterion

Proposition 149. *Let $\sum_{k\in\mathbb{Z}} c_k e^{ikt}$ be such that $c_k \geq 0$ and $c_k \searrow 0$ both as $k \to \infty$ and as $k \to -\infty$. Then $\sum_{k\in\mathbb{Z}} c_k e^{ikt}$ converges uniformly on $[\delta, 2\pi - \delta]$ for any $\delta \in (0, \pi)$. Consequently, the sum $f(t) = \sum_{k\in\mathbb{Z}} c_k e^{ikt}$ is continuous on $(0, 2\pi)$.*

Proof.

$\blacksquare$

Abel's Criterion can be used in this case even if c_k is not always positive. For instance, let $\sum_{k\in\mathbb{Z}}(-1)^k c_k e^{ikt}$ where the coefficient c_k are as in the statement of Proposition 149. What does the fact that

$$\left|\sum_{k\in\mathbb{Z}}(-1)^k(-1)^k e^{ikt}\right| = \left|\frac{1+(-1)^{n+1}e^{i(n+1)t}}{1-e^{it}}\right| \leq \frac{2}{|1+e^{it}|}$$

tell you?

These results also apply to the real part and the imaginary part of $\sum_{k\in\mathbb{Z}} c_k e^{ikt}$, i.e. to the series

$$a_0 + \sum_{k\geq 1} a_k \cos(kt) \quad \text{and} \quad \sum_{k\geq 1} b_k \sin(kt).$$

For instance, $\sum_{k\geq 1}\frac{\sin(kt)}{k}$ converges uniformly on $[\delta, 2\pi-\delta]$ for any $\delta > 0$. As a result, the sum is continuous on $(0, 2\pi)$.

However, even though $\sum_{k\geq 1}\frac{\sin(kt)}{k}$ converges for $t = 0$ and $t = 2\pi$, the function is not continuous on $[0, 2\pi]$ (see Exercise 9).

Let $T > 0$. A function $f : \mathbb{R} \to \mathbb{C}$ is $T-$**periodic** if $f(t + T) = f(t)$ for all $t \in \mathbb{R}$. The smallest positive T for which this holds is the period of the function.

Examples:

1. The functions $\cos$ and $\sin$ are $2\pi-$periodic.

2. The function $\tan$ is $\pi-$periodic.

3. The function defined by e^{ikt} is $\frac{2\pi}{k}-$periodic for any $k \in \mathbb{Z}$.

4. The function defined by e^{ikwt}, where $w = \frac{2\pi}{T}$ and $k \in \mathbb{Z}$, is $T-$periodic.

5. Let $f \in \mathcal{C}_c(\mathbb{R}, \mathbb{C})$, with **compact support** K (i.e. $f(t) = 0$ when $t \notin K$). Show that $\varphi_f : t \mapsto \sum_{k\in\mathbb{Z}} f(t-k)$ is $1-$periodic.

   Solution.

$\blacksquare$

If $f \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ is a $T-$periodic function, then f is bounded on the interval $[0, T]$, with

$$c_0(f) = \frac{1}{T} \int_0^T f(t)\, dt < \infty.$$

The complex number c_0 is the **mean value of** f, also given by

$$c_0(f) = \frac{1}{T} \int_a^{a+T} f(t)\, dt \quad \text{for all } a \in \mathbb{R}.$$

If $w = \frac{2\pi}{T}$ and $k \neq 0$, the function $g : t \mapsto e^{ikwt}$ is $T-$periodic. Then

$$c_0(g) = \frac{1}{T} \int_0^T e^{ikwt}\, dt = \frac{1}{T} \left[\frac{e^{ikwt}}{ikw} \right]_0^T = 0.$$

Hence, if $f(t) = \sum_{k\in\mathbb{Z}} c_k e^{ikwt}$ is uniformly convergent on $[0,T]$ and $T-$periodic, then

$$c_0(f) = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\int_0^T \left(\sum_{k\in\mathbb{Z}} c_k e^{ikwt}\right) dt = \sum_{k\in\mathbb{Z}} \frac{c_k}{T}\int_0^T e^{ikwt}\,dt = c_0$$

The sum and the integral can be interchanged because the series converges uniformly on $[0,T]$.

If $f \in \mathcal{C}(\mathbb{R},\mathbb{C})$ is $T-$periodic, the sequence $(c_k(f))$, where

$$c_k(f) = c_0\left(fe^{-ikwt}\right) = \frac{1}{T}\int_0^T f(t)e^{-ikwt}\,dt, \quad k \in \mathbb{Z},$$

is the sequence of **Fourier coefficients of** f.

If $w = \frac{2\pi}{T}$ and $f(t) = \sum_{k\in\mathbb{Z}} c_k e^{ikwt}$ is uniformly convergent on $[0,T]$, then $c_k(f) = c_k$.

Proposition 150. *The mapping $f \mapsto (c_k(f))_{k\in\mathbb{Z}}$ is a linear map from the vector space of continuous $T-$periodic functions to the space of bounded sequences indexed by $\mathbb{Z}$.*

More precisely,

$$\sup_{k\in\mathbb{Z}}\{|c_k(f)|\} \leq \|f\|_1 \leq \|f\|_\infty < \infty,$$

where $\|f\|_1 = \frac{1}{T}\int_0^T |f(t)|\,dt$.

Proof. ■

We can improve on Proposition 150 once we show that

$$\|f\|_2 = \left(\sum_{k\in\mathbb{Z}} |c_k(f)|^2\right)^{1/2}.$$

Proposition 151. *Let f be a $2\pi-$periodic function such that $f \in C^n$, $n > 0$. Then*

$$c_k(f) = \frac{1}{(ik)^n} c_k\left(f^{(n)}\right), \quad k \neq 0.$$

In particular,

$$|c_k(f)| \leq \frac{\|f^{(n)}\|_\infty}{|k|^n}$$

and so $|c_k(f)| \to 0$ as $|k| \to \infty$.

Proof.

■

As a corollary, if $f \in C^2$ is $2\pi-$periodic, then $\sum_{k\in\mathbb{Z}} c_k(f)e^{ikt}$ converges absolutely (and so uniformly) on $\mathbb{R}$.

The **Fourier series** of a $2\pi-$periodic function f is the series $\sum_{k\in\mathbb{Z}} c_k(f)e^{ikt}$; in this case, we write $f(t) \sim \sum_{k\in\mathbb{Z}} c_k(f)e^{ikt}$ (note that it is possible for f not to equal its Fourier series).

12.2.3 – Convergence of Fourier Series

The next results discuss the convergence of Fourier series.

Theorem 152. *Let f be $2\pi-$periodic. If $f \in C^2$, then the Fourier series $\sum_{k\in\mathbb{Z}} c_k(f)e^{ikt}$ converges absolutely (and so uniformly) to f on $\mathbb{R}$.*

Proof.

$\blacksquare$

The next result is a sufficient condition for a function to be equal to its Fourier series.

Theorem 153. *Let f be a continuous $2\pi-$periodic function such that*

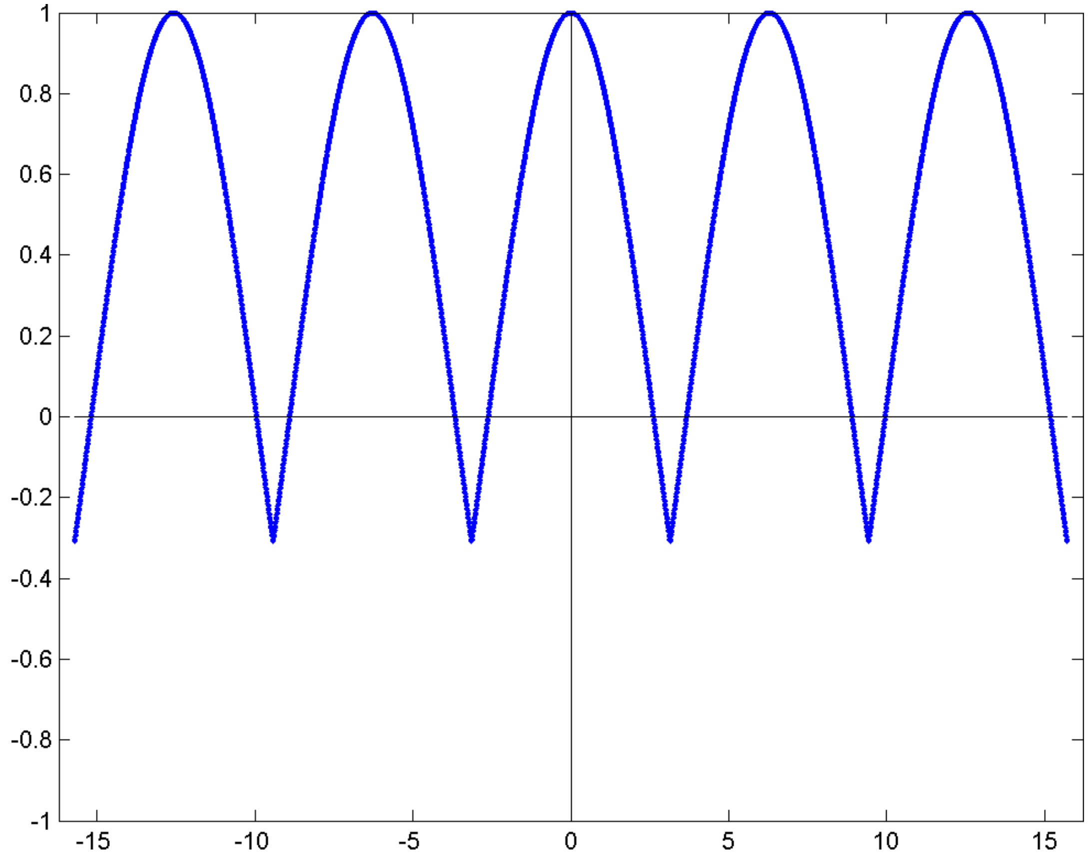
$$\sum_{k\in\mathbb{Z}} |c_k(f)| = M < \infty.$$

Then the Fourier series of f converges absolutely to f on $\mathbb{R}$ and is equal to f on $\mathbb{R}$.

Proof. ■

Example: Fix $a \in \mathbb{R}$ and let $f_a(t) = \cos(at)$, $|t| \leq \pi$. Extend f_a to $\mathbb{R}$ by periodicity. What is the Fourier series of f_a? Is it equal to f_a on $\mathbb{R}$?

Solution.



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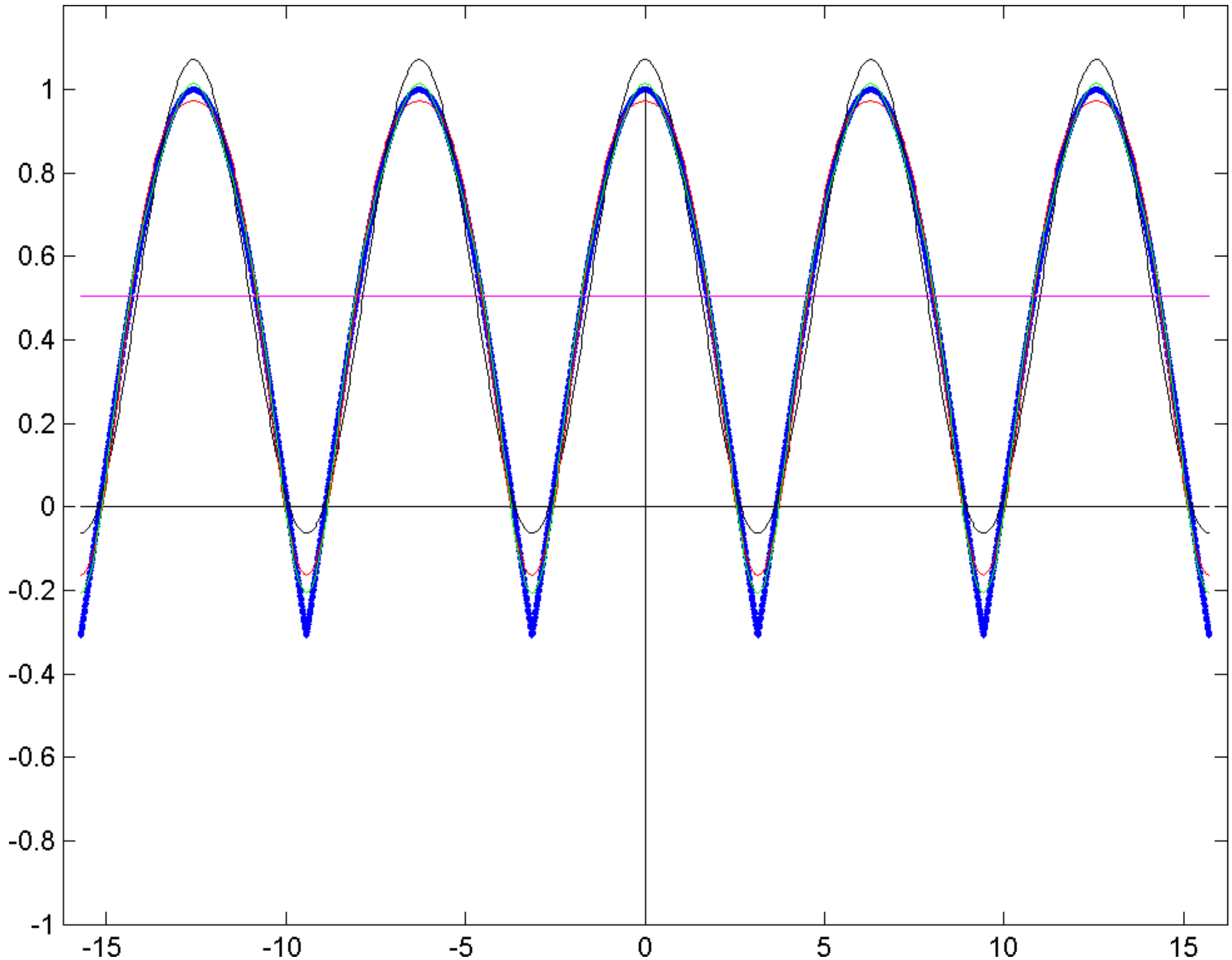
1
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0.6
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-0.2
-0.4
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-0.8
-1
-15
-10
-5
0
5
10
15

12.2.4 – Dirichlet's Convergence Theorem

Let $f:\mathbb{R}\to\mathbb{C}$ be a $2\pi-$periodic integrable function.

For $k\in\mathbb{Z}$, set

$$e_k(t)=e^{ikt}=\left(e^{it}\right)^k=(e_1(t))^k.$$

Let $N\in\mathbb{N}$. Define

$$S_N(f)(t):=\sum_{k=-N}^{N}c_k(f)e_k(t).$$

$S_N(f)$ is the partial sum of degree N for the Fourier series of f.

In what follows, we will write $\int := \frac{1}{2\pi}\int_0^{2\pi} = \frac{1}{2\pi}\int_a^{a+2\pi}$ for any $a \in \mathbb{R}$. We have

$$\begin{aligned} S_N(f)(t) &:= \sum_{k=-N}^{N} c_k(f)e_k(t) = \sum_{k=-N}^{N} e_k(t) \int f(y)e_k(-y)\,dy \\ &= \int f(y) \left\{ \sum_{k=-N}^{N} e_k(t)e_k(-y) \right\} dy \\ &= \int f(y) \left\{ \sum_{k=-N}^{N} e_k(t-y) \right\} dy \\ &= \int f(y)K_N(t-y)\,dy := (\hat{D}_N * f)(t), \end{aligned}$$

where, formally,

$$K_N(t) = \sum_{k=-N}^{N} e_k(t) = \sum_{k=-N}^{N} e^{ikt} = \frac{e^{-iNt} - e^{i(N+1)t}}{1 - e^{it}}$$

$$= \frac{1}{e^{iNt}} \left(\frac{1 - e^{i(2N+1)t}}{1 - e^{it}} \right) = \frac{\sin((N+1/2)t)}{\sin(t/2)}, \quad \text{when } t \notin 2\pi\mathbb{Z}.$$

Proposition 154. *The Dirichlet kernel is even,* $2\pi-$*periodic,* $c_0(K_N) = 1$*,* $\int_0^\pi K_N(t)\,dt = \pi$*, and*

$$K_N(0) = \lim_{t\to} K_N(t) = 2N + 1.$$

Proof. ■

Lemma 155. (Riemann-Lebesgue Lemma)
Let $f : [a, b] \to \mathbb{C}$ *be integrable over* $[a, b]$. *Then* $\lim_{n\to\infty} \int_a^b f(t)e^{int}\, dt = 0$.

Proof. ■

Theorem 156. (Dirichlet's Convergence Theorem)
Let $f : \mathbb{R} \to \mathbb{C}$ *be piecewise (with a finite number of discontinuities) and* $2\pi-$*periodic. If the following one-sided limits exist* $\forall x \in \mathbb{R}$:

$$f(x^{\pm}) = \lim_{h\searrow 0} f(x \pm h), \quad f'(x^{\pm}) = \lim_{h\searrow 0} \frac{f(x \pm h) - f(x)}{h},$$

then

$$S_N(f)(x) = \sum_{k=-N}^{N} c_k(f)e_k(x) \to \frac{f(x^+) + f(x^-)}{2}, \quad \text{as } N \to \infty.$$

Proof.

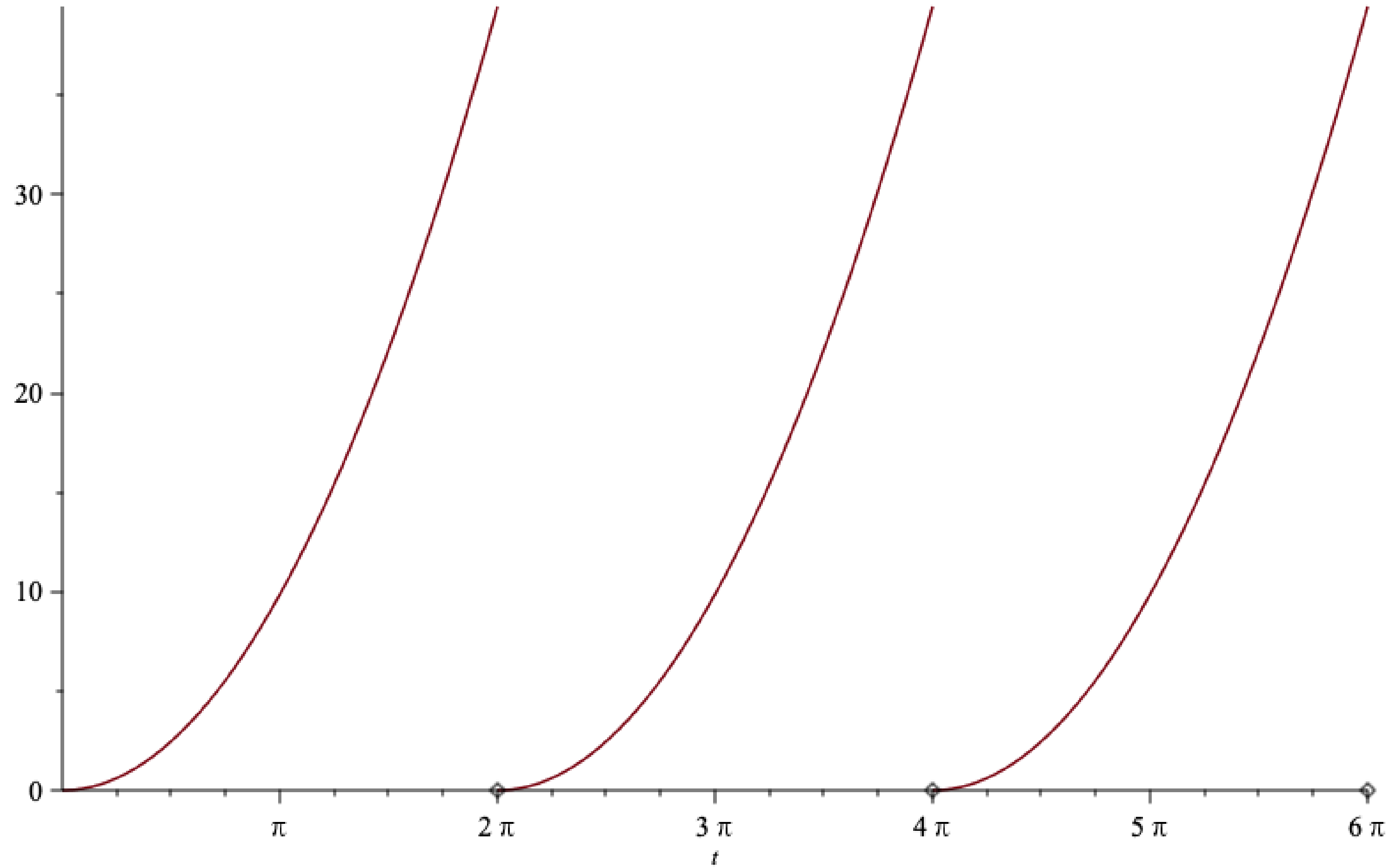
$\blacksquare$

In other words, if a periodic function f is "nice enough" (piecewise C^1), then it is equal to its Fourier series wherever f is continuous. At discontinuities of f, the Fourier series converges to the mean of the one-sided limits.

⚠ Some piecewise C^0 periodic functions have **divergent** Fourier series.

Example: Let $f : [0, 2\pi] \to \mathbb{R}$ be defined by $f(t) = t^2$. Extend f to $\mathbb{R}$ by periodicity. What is the Fourier series of f. Is it equal to f on $\mathbb{R}$?

Solution: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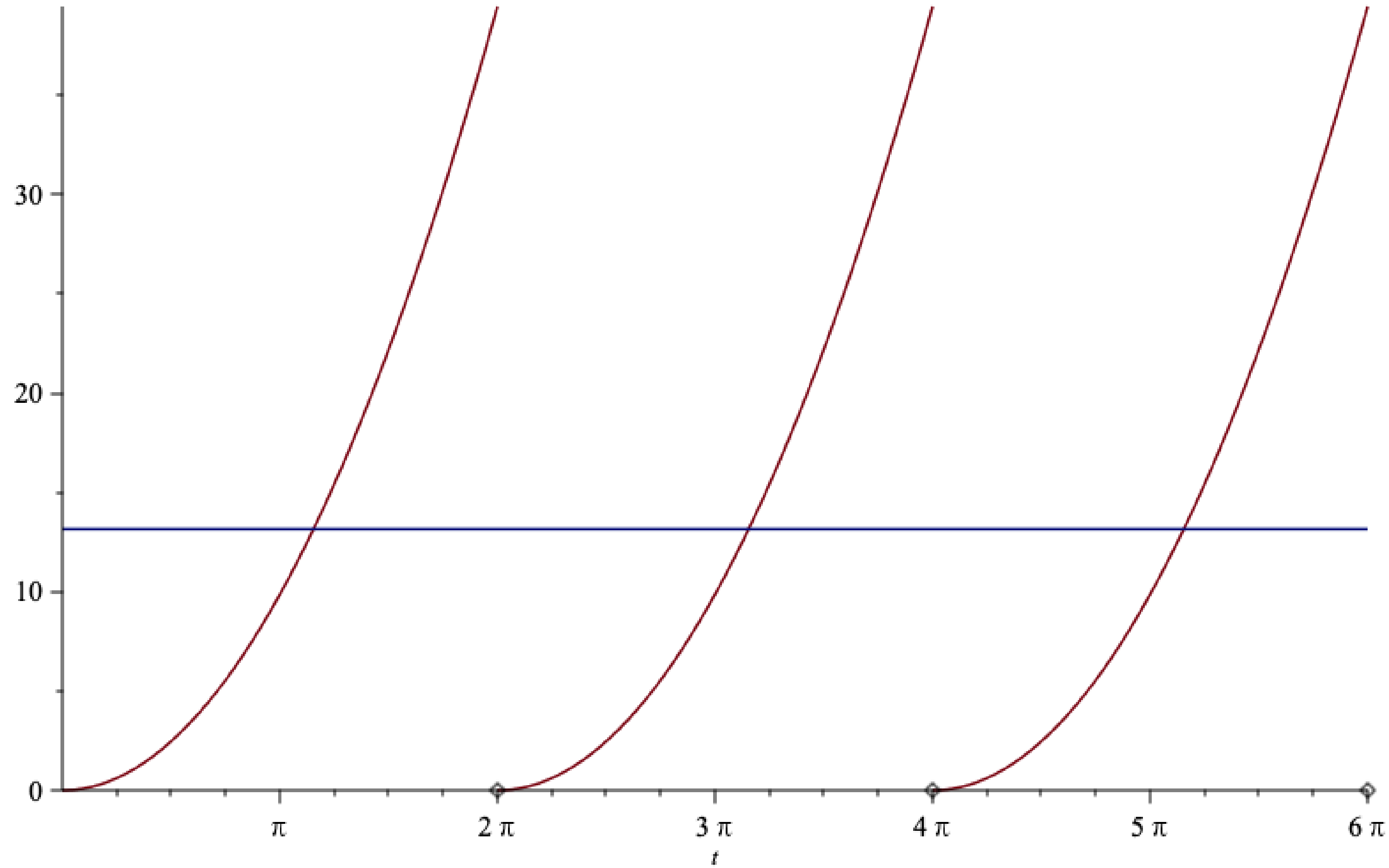
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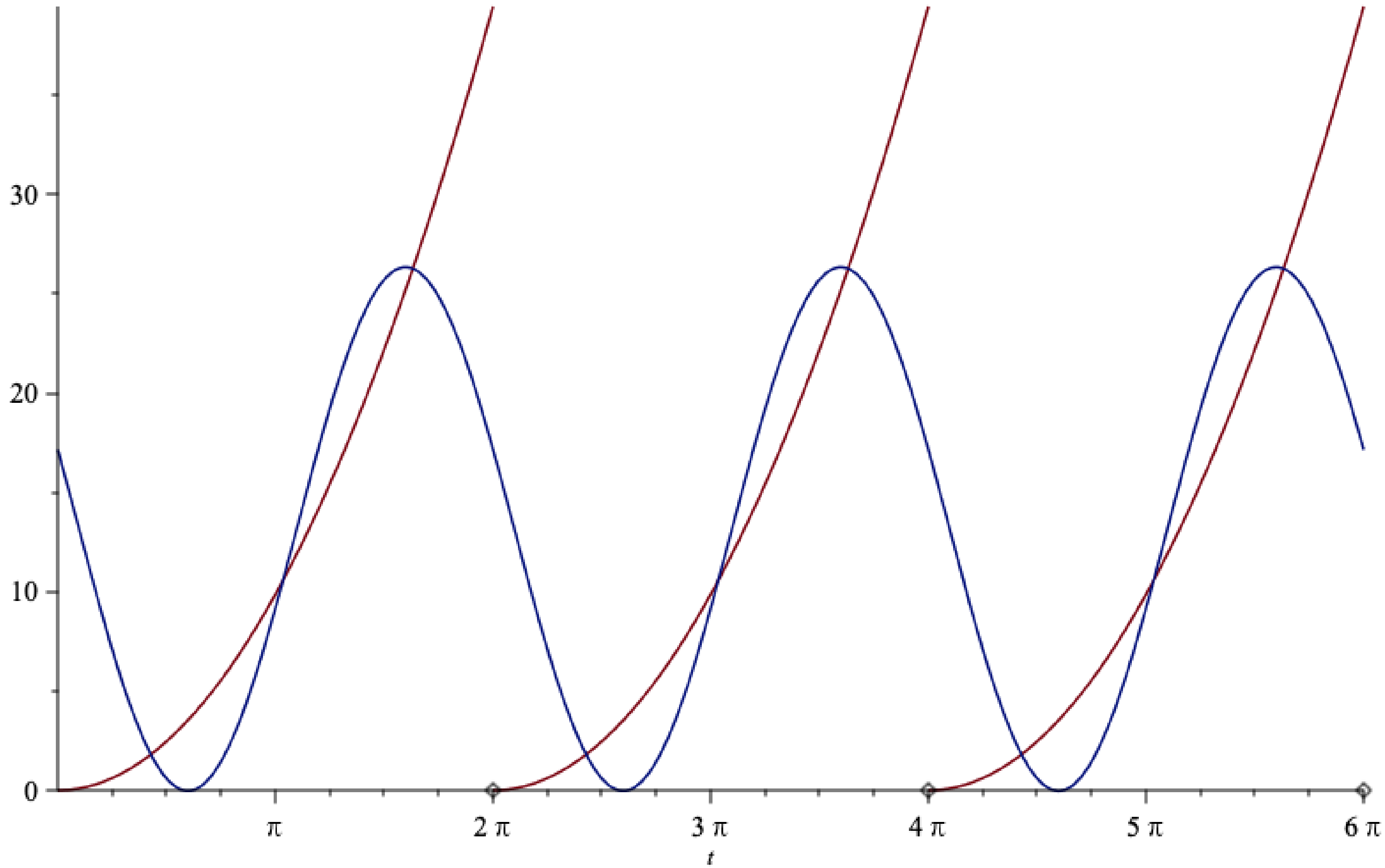
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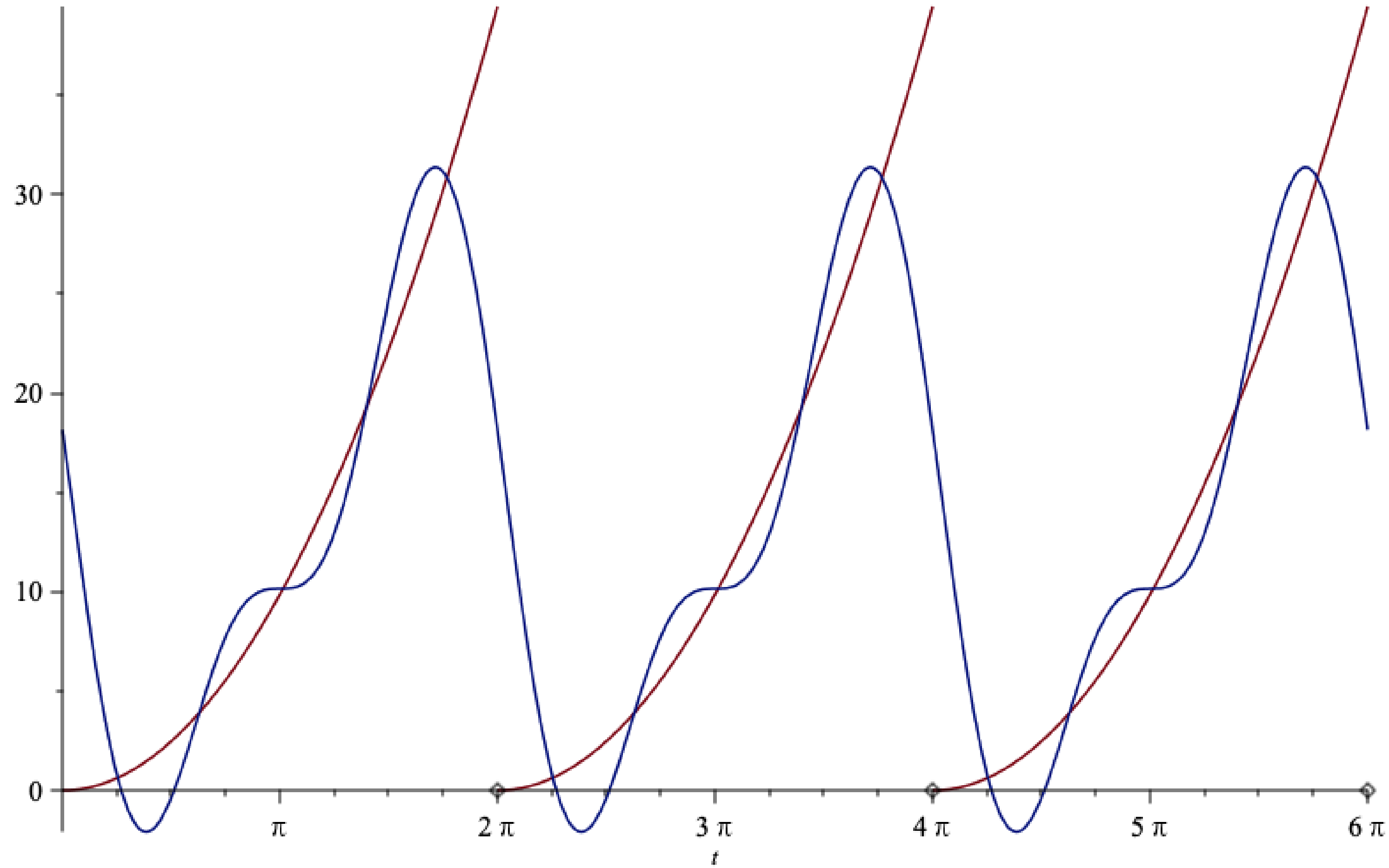
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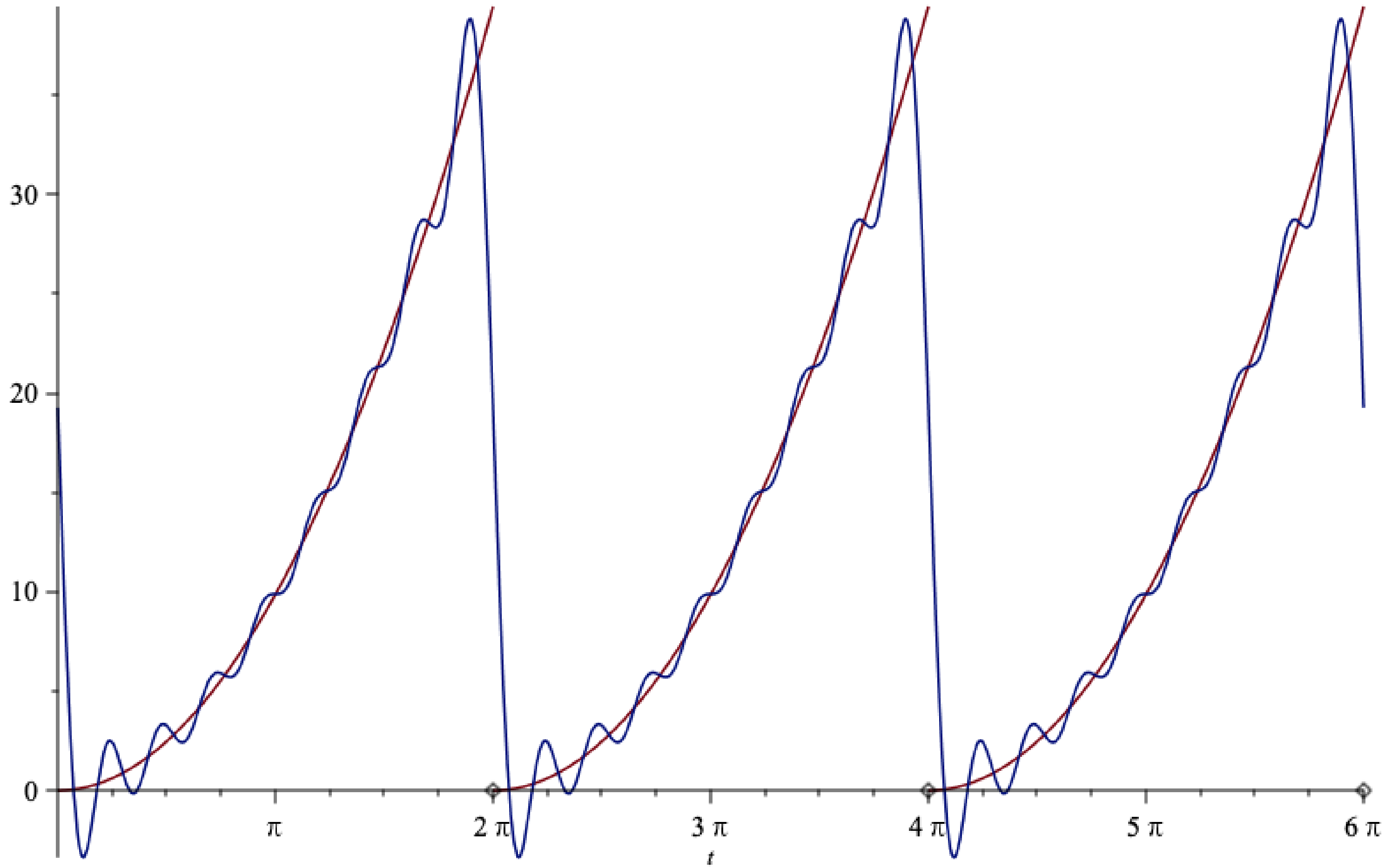
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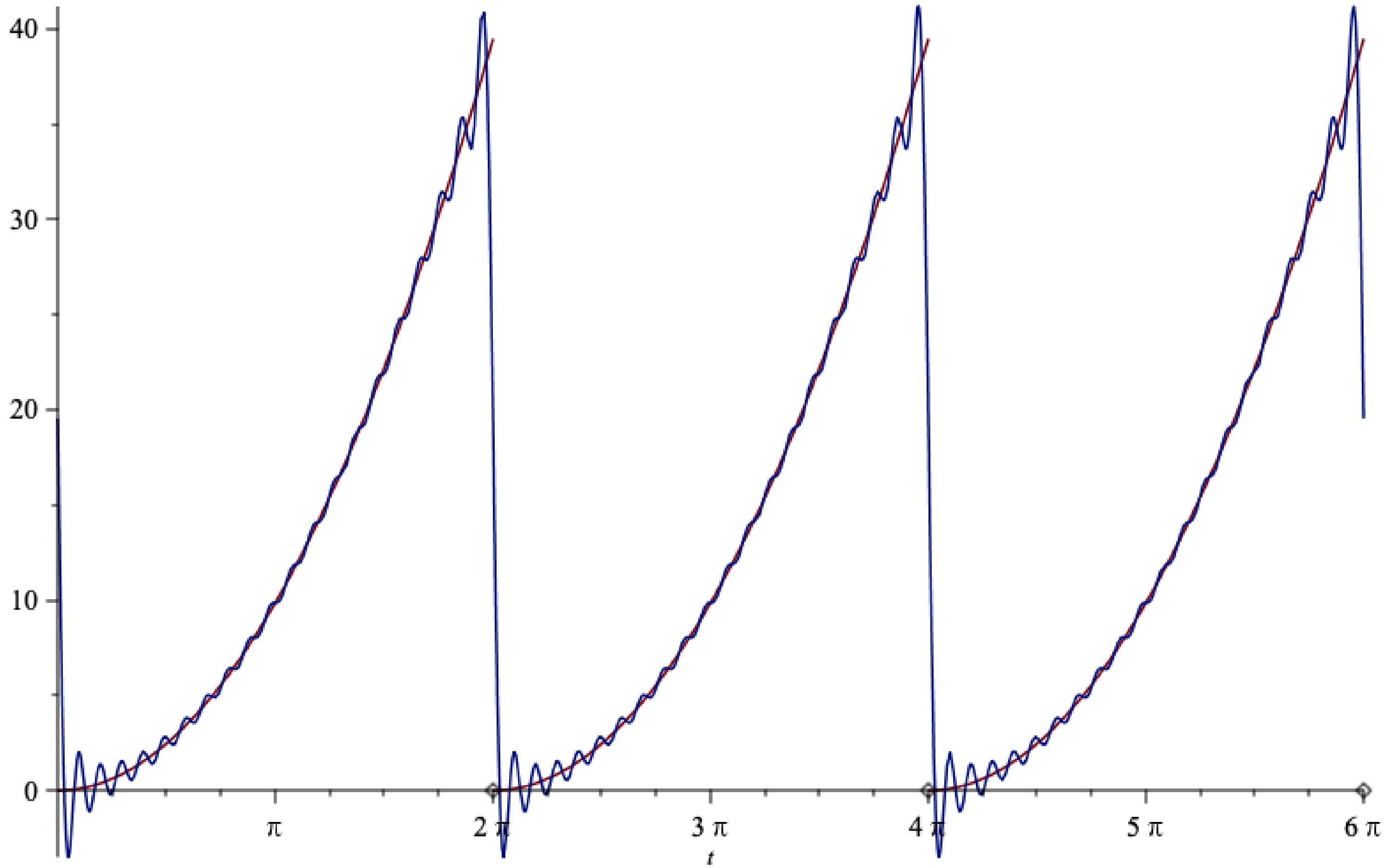
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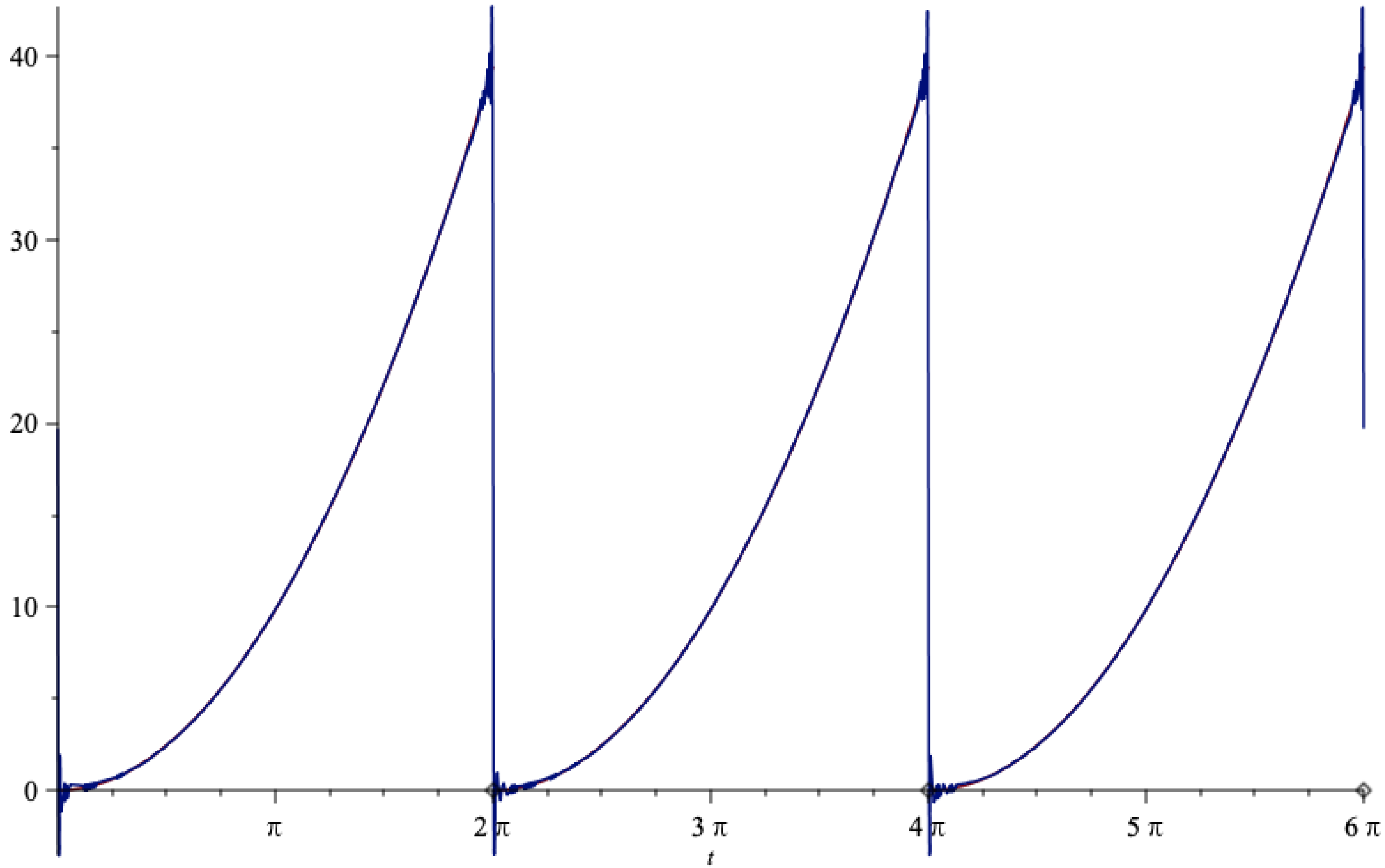
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Notice the overshooting of the partial sums as $t \to 2\pi\ell$, $\ell \in \mathbb{Z}$, which does not seem to dampen when $N \to \infty$.

This "universal" behaviour at discontinuities is termed **Gibbs' Phenomenon** (contrast the behaviour of the Fourier series of t^2 with that of $\cos(at)$ discussed earlier).

The explanation of the problem is linked with the $\limsup$ and $\liminf$ of the partial sums $S_n(f)(x_N)$ at points x_N that approach a discontinuity at x_0, but we will not discuss this any further.

12.2.5 – Quadratic Mean Convergence

The set of $2\pi-$periodic piecewise continuous functions from $\mathbb{R}$ to $\mathbb{C}$ is an **inner product space** together with

$$\langle f,g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt,$$

with **associated norm** $\|f\|_2 = \sqrt{\langle f,f\rangle}$.

Note that for $\mu,\nu\in\mathbb{Z}$, we have

$$\langle e_\mu,e_\nu\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i\mu t}e^{-i\nu t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{i(\mu-\nu)t}\,dt = \delta_{\mu,\nu} = \begin{cases}0, & \mu\neq\nu\\ 1, & \mu=\nu\end{cases}$$

For a given $N \in \mathbb{N}$ and a function f in the inner product space of the previous page, consider the partial sum

$$S_N(f) = \sum_{|k|\leq N} c_k(f) e_k(t).$$

For any $|k| \leq N$, we must have

$$c_k(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt}\, dt = \langle f, e_k \rangle.$$

But

$$\langle S_N(f), e_k \rangle = \sum_{|\ell|\leq N} c_\ell(f) \langle e_\ell, e_k \rangle = \sum_{|\ell|\leq N} c_\ell(f) \delta_{\ell,k} = c_k(f).$$

Thus, $\langle f - S_N(f), e_k\rangle = 0$ for all $|k| \leq N$ and we can write

$$f = S_N(f) + (f - S_N(f)),$$

with $S_N(f) \in \mathcal{P}_N = \text{Span}\{e_k\}_{|k|\leq N}$ and $f - S_N(f) \in \mathcal{P}_N^{\perp}$.

Note furthermore that since $\langle S_N, f - S_N(f)\rangle = 0$, then

$$\begin{aligned}
\|f\|_2^2 &= \langle f, f\rangle = \langle S_N(f) + (f - S_N(f)), S_N(f) + (f - S_N(f))\rangle \\
&= \langle S_N(f), S_N(f)\rangle + 2\text{Re}\underbrace{\langle S_N(f), f - S_N(f)\rangle}_{=0} + \langle f - S_N(f), f - S_N(f)\rangle \\
&= \|S_N(f)\|_2^2 + \|f - S_N(f)\|_2^2.
\end{aligned}$$

For any other function $g \in \mathcal{P}_N$, we see that

$$\begin{aligned}\|f-g\|^2 &= \| \underbrace{f - S_N(f)}_{\in \mathcal{P}_N^\perp} + \underbrace{S_N(f) - g}_{\in \mathcal{P}_N} \|_2^2 \\ &= \|f - S_N(f)\|_2^2 + \|S_N(f) - g\|_2^2 \geq \|f - S_N(f)\|_2^2.\end{aligned}$$

Since g was arbitrary,

$$\inf_{g \in \mathcal{P}_N} \|f-g\|_2^2 = \|f - S_N(f)\|_2^2 = \|f\|_2^2 - \|S_N(f)\|_2^2. \tag{1}$$

The partial sum $S_N(f)$ is thus the nearest trigonometric polynomial of $\mathcal{P}_N$ to f in the **quadratic mean**.

Theorem 157. (Parseval Identity)
Let f be a $2\pi-$periodic piecewise continuous function from $\mathbb{R}$ to $\mathbb{C}$. Then

$$\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^2\,dt=\sum_{k=-\infty}^{\infty}|c_k(f)|^2.$$

Proof.

$\blacksquare$

Parseval's Identity remains valid for functions that are locally integrable ($\int_K |f|\, dt < \infty$ for all $K \subseteq_K [0, 2\pi]$) instead of piecewise continuous.

The identity has multiple consequences: since it (also) applies (also) to any locally integrable $2\pi-$periodic function $f : \mathbb{R} \to \mathbb{C}$, the series

$$\sum_{k \in \mathbb{Z}} |c_k(f)|^2$$

converges, which shows that $|c_k(f)|^2 \to 0$, and thus $c_k(f) \to 0$ as $k \to \pm\infty$ (Riemann-Lebesgue Lemma).

It can also be used to show that any $2\pi-$periodic continuous function $f : \mathbb{R} \to \mathbb{C}$ whose Fourier series converges uniformly on $\mathbb{R}$ must be equal to said series (compare with Dirichlet's Convergence Theorem).

12.3 – Exercises

1. Let (g_n) be a sequence of functions. Show that $\sum g_n$ converges absolutely if and only if $\exists (a_n) \subseteq \mathbb{R}^+$ such that $\sum a_n$ converges and $\|g_n\|_\infty \leq a_n$ for all n. Use that result to show that the series of functions $\sum g_n$, where $g_n : [0,1] \to \mathbb{R}$ is defined by $g_n(x) = \frac{x^n}{n^2}$, is absolutely convergent on $[0,1]$.
2. For each of the theorems of Section 12.1.1 (except for Theorem 144), find an example showing that the result does not hold if uniform convergence is replaced by pointwise convergence.
3. Prove Theorem 144.
4. Find some examples showing that the result of Theorem 144 does not hold in general if absolute convergence is replaced by a weaker type of convergence.
5. Let $g_n : \mathbb{R} \to \mathbb{R}$ be defined by $g_n(x) = \frac{x^n}{n!}$ for each $n \in \mathbb{N}$. Show that each of the following series of functions converges absolutely on $\mathbb{R}$.
   (a) $S = \sum (-1)^{n+1} g_{2n+1}$
   (b) $C = \sum (-1)^n g_{2n}$
   (c) $E = \sum g_n$
6. Let S, C, E be as in the previous question. Using the appropriate theorems, show that for any $x \in \mathbb{R}$ show that
$$S'(x) = C(x), \quad C'(x) = -S(x), \quad E'(x) = E(x).$$

7. Find examples showing that the three conditions in the statement of Proposition 147 are independent from one another.
8. Prove Proposition 148.
9. Show that the function $f : [0, 2\pi] \to \mathbb{R}$ defined by $f(t) = \sum_{k\geq 1} \frac{\sin(kt)}{k}$ is not continuous on $[0, 2\pi]$.
10. Prove Theorem 153.
11. Using the Fourier series of the cosine, show that $\pi \cot(a\pi) = \sum_{k\in\mathbb{Z}} \frac{a}{a^2-k^2}$ for all $a \notin \mathbb{Z}$ (also known as **Euler's Formula**).
12. Prove the properties of the Dirichlet kernel (Proposition 154).
13. Show that $\langle f, g\rangle$ (see page 73) defines an inner product on the set of $2\pi-$periodic piecewise continuous functions from $\mathbb{R}$ to $\mathbb{C}$.
14. Prove the Riemann-Lebegue Lemma without using Parseval's Identity.
15. Show that any $2\pi-$periodic continuous function $f : \mathbb{R} \to \mathbb{C}$ whose Fourier coefficients are all 0 must be the zero function.
16. Let $(a_n) \subseteq \mathbb{C}$ be such that $a_n \to \ell$ and let $(\varepsilon_n) \subseteq \mathbb{R}^+$ be a divergent sequence. Define a sequence $(b_n) \subseteq \mathbb{C}$ by
$$b_n = \frac{\sum_{i=1}^n a_i\varepsilon_i}{\sum_{i=1}^n \varepsilon_i}.$$
Show that $b_n \to \ell$.

17. (a) Let (f_n) be the sequence of functions defined by

$$f_n : \mathbb{R}_0^+ \to \mathbb{R}, \quad f_n(x) = \begin{cases} \left(1 - \frac{x}{n}\right)^n & x \in [0, n] \\ 0 & x > n \end{cases}$$

Show that $f_n \rightrightarrows f$ on $\mathbb{R}_0^+$, where $f : \mathbb{R}_0^+ \to \mathbb{R}$ is defined by $f(x) = e^{-x}$.

(b) Let $U \subseteq_K \mathbb{C}$ and let (f_n) be the sequence of functions defined by

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = \left(1 + \frac{z}{n}\right)^n.$$

Show that $f_n \rightrightarrows f$ on K, where $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(z) = e^z$.

18. For any $n \in \mathbb{N}^\times$, let $u_n : \mathbb{R}_0^+ \to \mathbb{R}$ be defined by $u(x) = \frac{x}{n^2+x^2}$.

(a) Show that $\sum u_n \to f$ for some $f \in \mathcal{C}(\mathbb{R}_0^+, \mathbb{R})$, but that $\sum u_n \not\rightrightarrows f$ on $\mathbb{R}_0^+$.

(b) Show that $\sum (-1)^n u_n \rightrightarrows g$ on $\mathbb{R}_0^+$ for some $g \in \mathcal{C}(\mathbb{R}_0^+, \mathbb{R})$, but that $\sum (-1)^n u_n$ is not absolutely convergent on $\mathbb{R}_0^+$.

19. What can you say about a function $f : \mathbb{R} \to \mathbb{R}$ which is the uniform limit of a sequence of polynomials (P_n)?

20. Consider the sequence of functions $(f_n) \subseteq \mathcal{C}([0, \pi/2], \mathbb{R})$ defined by $f_n(x) = \cos^n x \sin x$ for all $n \in \mathbb{N}$.
    (a) Let $\mathcal{O} : [0, \pi/2] \to \mathbb{R}$ be the zero function. Show that $f_n \rightrightarrows \mathcal{O}$ on $[0, \pi/2]$.
    (b) Consider the sequence of functions (g_n) defined by $g_n = (n+1)f_n$. Let $\delta > 0$. Show that $g_n \rightrightarrows \mathcal{O}$ on $[\delta, \pi/2]$ but that
$$\int_0^{\pi/2} g_n(t)\,dt \not\to 0.$$
21. Theses results are due to Dini.
    (a) Let $(f_n) \in \mathcal{C}([a, b], \mathbb{R})$ be an increasing sequence of functions (i.e. for all $x \in [a, b]$ and for all $n \in \mathbb{N}$, we have $f_n(x) \leq f_{n+1}(x)$). If $f_n \to f$ on $[a, b]$ where $f \in \mathcal{C}([a, b], \mathbb{R})$, show that $f_n \rightrightarrows f$ on $[a, b]$.
    (b) Let $(f_n) \in \mathcal{C}([a, b], \mathbb{R})$ be a sequence of increasing functions (i.e. for all $x \geq y \in [a, b]$ and for all $n \in \mathbb{N}$, we have $f_n(x) \geq f_n(y)$). If $f_n \to f$ on $[a, b]$ where $f \in \mathcal{C}([a, b], \mathbb{R})$, show that $f_n \rightrightarrows f$ on $[a, b]$.
22. Determine whether $\sum \mathbf{x}_n$ converges in $(\mathbb{R}^2, \|\cdot\|_2)$, where
$$\mathbf{x}_n = \left(\frac{(\sin n)^n}{n^2}, \frac{1}{n^2} \right).$$

    If so, does $\sum \mathbf{x}_n$ converge absolutely?

23. Compute the values of the following convergent series

$$\sum_{n=1}^{\infty}\frac{1}{n^2},\quad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2},\quad \sum_{n=1}^{\infty}\frac{1}{n^4},$$

using the $2\pi-$periodic function defined by $f(x)=1-x^2/\pi^2$ over the interval $[-\pi,\pi]$.