Mathematical Analysis

Chapter 2
The Real Numbers

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Overview

In a course on real analysis, the fundamental object of study is the set of real numbers.

In this chapter, we

- introduce $\mathbb{R}$ and some of its important properties,
- discuss the cardinality of sets, and
- provide a first analytical result, whose proof will serve as an introduction to the discipline.

Outline

2.1 – Hierarchy of Number Systems

In this first course, **analysis** is a theory on real numbers $\mathbb{R}$, that is, the objects with which we work are **real numbers**, **real sets**, and **real functions**.

We will see at a later stage that we can conduct analysis on any **metric space** (such as $\mathbb{R}^n$ and $\mathbb{C}$, for instance).

There is a natural hierarchy amongst number sets, which you have no doubt encountered in your courses:

$$\mathbb{N}^\times \subsetneq \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{A} \subsetneq \mathbb{R} \subsetneq \mathbb{C}.$$

The **positive integers** $\mathbb{N}^\times$ are the counting numbers; **zero** is added to $\mathbb{N}^\times$ to form $\mathbb{N}$, in which all equations $x + a = b$, $b \geq a \in \mathbb{N}^\times$ have a solution.

Similarly, the **integers** $\mathbb{Z}$ are built by adding new numbers to $\mathbb{N}$ in order for all equations of the form $x + a = b$, $a, b \in \mathbb{N}$ to have solutions.

For the **rational numbers** $\mathbb{Q}$, the equations in question have the form $ax + b = 0$, $a, b \in \mathbb{Z}$, $b \neq 0$.

For the **algebraic numbers** $\mathbb{A}$, we are looking at equations of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0, \quad a_i \in \mathbb{Q},$$

and for **complex numbers** $\mathbb{C}$, equations of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0, \quad a_i \in \mathbb{R}.$$

In other words, number sets are generally easy to construct once we have the right building blocks... except when it comes to the **real numbers** $\mathbb{R}$.

In this chapter and the next, we will introduce concepts that will allow us to **formally define** $\mathbb{R}$.

In what follows, we will make use of the following axiom about the set $\mathbb{N}$.

Axiom. (Well-Ordering Principle)
Any non-empty subset of $\mathbb{N}$ *has a smallest element.*

We shall discuss how to define the "smallest" element of a set momentarily. We shall also discuss how to measure the "size" of a set in Section 2.2: for the moment, we will leave you with the following tantalizing remark: $\mathbb{Q}$ is infinite, but **it contains infinitely more holes than it does elements**.

2.1.1 – Field and Order Properties of $\mathbb{R}$; Completeness

A **field** F is a set endowed with two binary operations: an **addition**

$$+ : F \times F \to F, \quad +(a,b) = a + b$$

and a **multiplication**

$$\cdot : F \times F \to F, \quad \cdot(a,b) = ab,$$

which satisfy the 9 **field properties**:

(A1) **commutativity of $+$:** $\forall a, b \in F$, $a + b = b + a$;

(A2) **associativity of $+$:** $\forall a, b, c \in F$, $(a + b) + c = a + (b + c)$;

(A3) **existence of neutral element for $+$:** $\exists 0 \in F$, $\forall a \in F$, $a + 0 = a$;

(A4) **inverse with respect to $+$:** $\forall a \in F$, $\exists! b \in F$, $a + b = 0$;

(M1) **commutativity of $\cdot$:** $\forall a, b \in F$, $ab = ba$

(M2) **associativity of $\cdot$:** $\forall a, b, c \in F$, $(ab)c = a(bc)$

(M3) **existence of neutral element for $\cdot$:** $\exists 1 \in F$, $\forall a \in F$, $1a = a$

(M4) **inverse with respect to $\cdot$:** $\forall a \in F^{\times}$, $\exists! b \in F$, $ab = 1$

(D1) **distributivity of $\cdot$ over $+$:** $\forall a, b, c \in F$, $a(b + c) = ab + ac$

Examples:

An **order** on a set F is a binary relation "$<$" satisfying the **order properties**:

(O1) **trichotomy:** $\forall a, b, c \in F$, $a < b$ or $a = b$ or $b < a$;

(O2) **transitivity:** $\forall a, b, c \in F$, if $a < b$ and $b < c$, then $a < c$.

(O3) $\forall a, b, c \in F$, if $a < b$, then $a + c < b + c$.

(O4) **(specific to $\mathbb{R}$):** $\forall a, b, c \in \mathbb{R}$, if $a < b$ and $c > 0$, then $ac < bc$.

Examples:

1.

2.

Let $(F, <)$ be an ordered set and $S \subseteq F$. If $a < b$ or $a = b$, we write $a \leq b$.

The element $u \in F$ is an **upper bound of** S if $s \leq u$ for all $s \in S$. In that case, we say that S is **bounded above**.

If u is the smallest upper bound of S, we say that it is the **supremum** of S, denoted $u = \sup S$.

The element $v \in F$ is a **lower bound of** S if $v \leq s$ for all $s \in S$. In that case, we say that S is **bounded below**.

If v is the largest lower bound of S, we say that it is the **infimum** of S, denoted $u = \inf S$.

If the set S is bounded both above and below, we say that it is **bounded**.

Example: If $S = \{x \in \mathbb{Q} \mid 2 < x < 3\}$, then $\inf S = 2$.

Proof.

$\blacksquare$

This "proof" rests on thin ice: it assumes that

1. the infimum exists in the first place;
2. if the infimum exists, it is a rational number, and
3. a rational number can be found between any two distinct rationals.

These are valid **in this specific case**, but not in general. More on this later.

Example: If $S = \mathbb{N}$, then $\inf S = 1$.

Proof.

$\blacksquare$

A set $(F,<)$ is **complete** if any non-empty bounded subset $S \subseteq F$ has a supremum and an infimum.

Example: $\mathbb{Q}$ is not complete.

Proof.

$\blacksquare$

The set $\mathbb{R}$ of **real numbers** is the smallest complete ordered field containing $\mathbb{N}$, with order $a < b \Longleftrightarrow b - a > 0$.

2.1.2 – Archimedean Property

Classically, $\mathbb{R}$ is constructed using **Dedekind cuts** or **Cauchy sequences**: in effect, $\mathbb{R}$ is constructed by "filling the holes" of $\mathbb{Q}$.

We will discuss Cauchy sequences in Chapter 3 and provide the outline of $\mathbb{R}$'s construction in an interlude.

For now, we assume that $\mathbb{R}$ is available and that is satisfies the properties mentioned previously.

The course's first result seems intuitively "obvious" but its proof is not.

Theorem 1. (Archimedean Property)
Let $x \in \mathbb{R}$. Then $\exists n_x \in \mathbb{N}^\times$ such that $x < n_x$.

Proof.

∎

Example: Show that $\inf\{\frac{1}{n} \mid n \in \mathbb{N}^\times\} = 0$.

Proof.

■

Theorem 2. (Variants of the Archimedean Property)
Let $x, y \in \mathbb{R}^+$. Then $\exists n_1, n_2, n_3 \geq 1$ such that

1. $x < n_1 y$;
2. $0 < \frac{1}{n_2} < y$, *and*
3. $n_3 - 1 \leq x < n_3$.

Proof.

1.

2.

3.

There are other variants, but these are the ones we will use the most. ∎

It is thus always possible to find an integer greater than any specified real number. This result is extremely useful – we use it next to show the existence of **irrational numbers**.

Corollary. The positive root of $x^2 = 2$ lies in $\mathbb{R}$ but not in $\mathbb{Q}$.

Proof.

$\blacksquare$

From this point on, when we mention the Archimedean Property, we mean one of the four variants from Theorems 1 and 2.

2.1.3 – Absolute Value and Useful Inequalities

The real numbers enjoy another set of useful and interesting properties.

Theorem 3. (Bernoulli's Inequality)
Let $x \geq -1$. *Then* $(1+x)^n \geq 1 + nx$, $\forall n \in \mathbb{N}$.

Proof.

$\blacksquare$

Note: at first glance, it might appear that we did not use the hypothesis that $x \geq -1$. But the assumption is essential – if $1 + x < 0$, the use of the Induction Hypothesis in the string of inqualities is invalid.

Theorem 4. (CAUCHY'S INEQUALITY)
If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, then

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right).$$

(The indices are understood to run from 1 to n in what follows.) Furthermore, if $b_j \neq 0$ for one of $1 \leq j \leq n$, then equality holds if and only if $\exists s \in \mathbb{R}$ such that $a_i = sb_i$ for all $i = 1, \ldots, n$.

Proof.

■

Theorem 5. $(\textsc{Triangle Inequality})$ *If* $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$,

$$\left(\sum (a_i + b_i)^2\right)^{1/2} \leq \left(\sum a_i^2\right)^{1/2} + \left(\sum b_i^2\right)^{1/2}.$$

Furthermore, if $b_j \neq 0$ *for one of* $1 \leq j \leq n$, *then equality holds if and only if* $\exists s \in \mathbb{R}$ *such that* $a_i = sb_i$ *for all* $i = 1, \ldots, n$.

Proof.

$\blacksquare$

In the Triangle Inequality, if we set $n = 1$, we obtain the very useful inequality:

$$\sqrt{(a+b)^2} \leq \sqrt{a^2} + \sqrt{b^2},$$

which we usually write $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

The function $|\cdot| : \mathbb{R} \to \mathbb{R}$ is the **absolute value**, which can be used to represent the distance between a real number and the origin.

It is defined by

$$|x| = \begin{cases} x, & x \geq 0 \\ x, & x \leq 0 \end{cases}$$

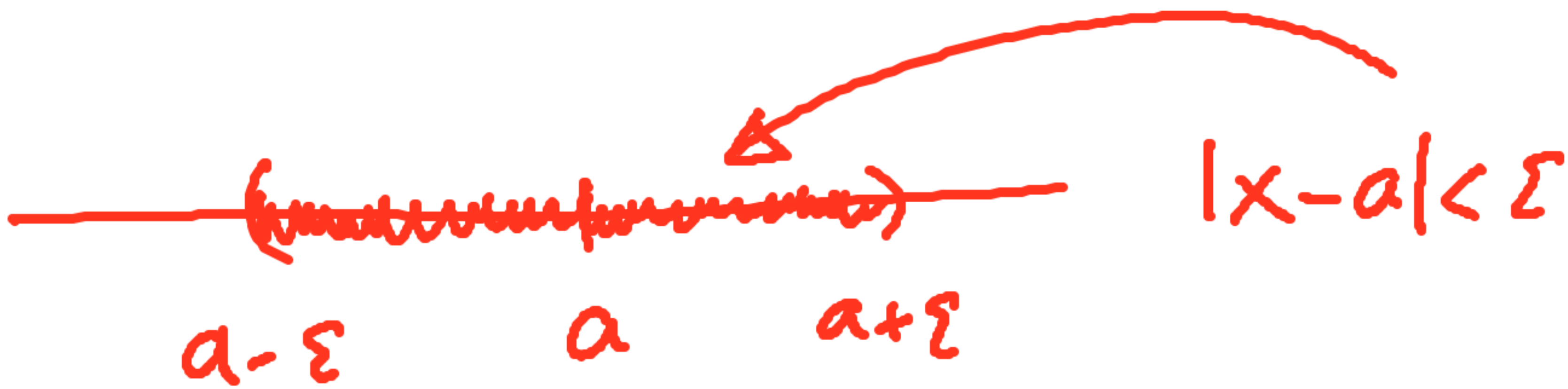
Equipped with this function, $\mathbb{R}$ is an example of a **normed space**. Normed space will be discussed at a later stage.

Theorem 6. (Properties of the Absolute Value)
If $x, y \in \mathbb{R}$, then

1. $|x| = \sqrt{x^2}$
2. $-|x| \leq x \leq |x|$
3. $|xy| = |x||y|$
4. $|x + y| \leq |x| + |y|$
5. $|x - y| \leq |x| + |y|$
6. $||x| - |y|| \leq |x - y|$

Remark: the following inequality will play a central role in the chapters to come:

$$|x-a|<\varepsilon \Longleftrightarrow a-\varepsilon<x<a+\varepsilon.$$



We finish this section with an intriguing result about the distribution of rationals and irrationals among the reals.

2.1.4 – Density of $\mathbb{Q}$

Theorem 7. (Density of $\mathbb{Q}$)
Let $x, y \in \mathbb{R}$ be such that $x < y$. Then, $\exists r \in \mathbb{Q}$ such that $x < r < y$.

Proof.

1.

2.

3.

$\blacksquare$

Corollary. Let $x, y \in \mathbb{R}$ with $x < y$. Then, $\exists z \notin \mathbb{Q}$ such that $x < z < y$.

Proof.

$\blacksquare$

It is thus possible to find rationals and irrationals between any two real numbers $x < y$. In spite of this, $\mathbb{Q}$ is much "smaller" than $\mathbb{R} \setminus \mathbb{Q}$.

2.2 – Cardinality of Sets

In the set hierarchy $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$, the first three sets are of the same size, while the last one is "infinitely" larger.

For all $n \in \mathbb{N}^{\times}$, define the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$.

A set S is **finite** if $S = \varnothing$ or if there exists a bijection $f : \mathbb{N}_n \to S$ for some $n \in \mathbb{N}^{\times}$. If S is not finite, it is **infinite**.

If S is infinite and there exists a bijection $f : \mathbb{N} \to S$, then S is **countable**. Otherwise, it is **uncountable**.

Note: in some references, finite sets are called **finitely countable** sets, and countable sets are called **infinitely countable** sets.

Consider two sets S_n and T_n, both with n distinct elements:

$$S_n = \{s_1, \ldots, s_n\}, \quad T_n = \{t_1, \ldots, t_n\}.$$

These two finite sets have the same size: there is a bijection $f : S_n \to T_n$, $f(s_i) = t_i$ for $1 \leq i \leq n$ (it is not the only such bijection).

In general, two sets S, T are said to have the same **cardinality**, denoted $|S| = |T|$, if there exists a bijection $f : S \to T$.

If S, T are finite, $|S| = |T|$ means that the two sets **have the same number of elements**: $|S| = |T| = |\mathbb{N}_n| = n$ for some $n \in \mathbb{N}$.

If S, T are infinite, the "number of elements" is not a well-defined, which can lead to counter-intuitive results.

Examples:

1.

2.

So two sets can have equal cardinality even when one is strictly contained in the other (this can only happen with infinite sets, however).

Theorem 8. *If S is an infinite subset of a countable set A, then S is countable.*

Proof.

■

General Remark: if you find it difficult to follow a proof, it is never a bad idea to try it with specific examples satisfying the hypotheses.

⚠ If you have to give a proof, an example only works if you are trying to show that some statement is **false**. A direct proof **never** uses examples.

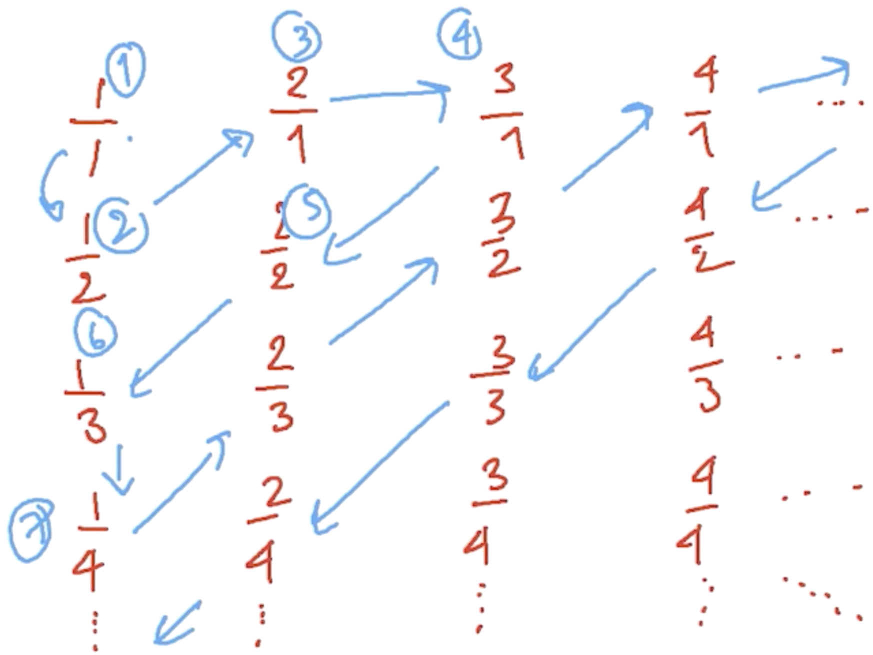
The contrapositive of Theorem 8 gives a useful way to show that a set is uncountable: if $S \subseteq A$ is uncountable, then A is uncountable.

2.2.1 – Cardinality of $\mathbb{Q}$

Another way to think of countable sets is that they could be enumerated, at least conceptually, in an infinite list.

Theorem 9. *The set $\mathbb{Q}$ is countable.*

Proof.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1/2 2/2 3/2 4/2
1/3 2/3 3/3 4/3
1/4 2/4 3/4 4/4
1 2 3 4 5 6 7

■

2.2.2 – Cardinality of $\mathbb{R}$

We now show that a set which would seem to be much smaller than $\mathbb{Q}$ at a first glance is in fact much larger than $\mathbb{Q}$ from a cardinality perspective, using the celebrated **Cantor diagonal argument**.

Theorem 10. *The set $I = [0, 1]$ is uncountable.*

Proof.

$\blacksquare$

Since $[0,1] \subseteq \mathbb{R}$, then $\mathbb{R}$ is also uncountable. What about $\mathbb{R} \setminus \mathbb{Q}$?

In general, is it possible for the union of two countable sets to be uncountable? Is the intersection of two uncountable sets uncountable?

2.3 – Nested Intervals Theorem

We end this chapter with an important result concerning nested intervals. In style and rigour, its proof is representative of analytical reasoning.

Theorem 11. (NESTED INTERVALS)
For every integer $n \geq 1$, let $[a_n, b_n] = I_n$ be such that

$$I_1 \supseteq I_2 \supseteq \cdots I_n \supseteq I_{n+1} \supseteq \cdots$$

Then there exists $\psi, \eta \in \mathbb{R}$ such that $\psi \leq \eta$ and $\bigcap_{n \geq 1} I_n = [\psi, \eta]$.

Furthermore, if $\inf\{b_n - a_n \mid n \in \mathbb{N}\} = 0$, then $\psi = \eta$.

Proof.

$\blacksquare$

Why can we conclude that $\eta - \psi = 0$ if $0 \leq \eta - \psi < \varepsilon$ for all $\varepsilon > 0$?

In general, if $a \leq x < a + \varepsilon$ for all $\varepsilon > 0$, then $x = a$. If $x \neq a$, $\exists \delta > 0$ such that $x = a + \delta$. Thus, if $\varepsilon = \delta$, which is possible since ε can take on any positive value, we would have $\delta = x - a < \varepsilon = \delta$, a contradiction.

Example:

⚠ We can only use a theorem if the hypotheses are satisfied (even though the conclusion may hold nonetheless). The intervals $I_n = (1 - \frac{1}{n}, 1 + \frac{1}{n})$, $n \geq 1$ are such that their intersection is $\{1\}$, but not because of the NVT.

2.4 – Exercises

1. Let $a, b \in \mathbb{R}$ and suppose that $a \leq b + \varepsilon$ for all $\varepsilon > 0$. Show that $a \leq b$.
2. Let $c > 0$ be a real number.
   (a) If $c > 1$, show that $c^n \geq c$ for all $n \in \mathbb{N}$ and that $c^n > 1$ if $n > 1$.
   (b) If $0 < c < 1$, show that $c^n \leq c$ for all $n \in \mathbb{N}$ and that $c^n < 1$ if $n > 1$.
3. Let $c > 0$ be a real number.
   (a) If $c > 1$ and $m, n \in \mathbb{N}$, show that $c^m > c^n$ if and only if $m > n$.
   (b) If $0 < c < 1$ and $m, n \in \mathbb{N}$, show that $c^m > c^n$ if and only if $m < n$.
4. Let $S_2 = \{x \in \mathbb{R} \mid x > 0\}$. Does S_2 have lower bounds? Does S_2 have upper bounds? Does $\inf S_2$ exist? Does $\sup S_2$ exist? Prove your statements.
5. Let $S_4 = \left\{1 - \frac{(-1)^n}{n} \mid n \in \mathbb{N}\right\}$. Find $\inf S_4$ and $\sup S_4$.
6. Let $S \subseteq \mathbb{R}$ be non-empty. Show that if $u = \sup S$ exists, then for every number $n \in \mathbb{N}$ the number $u - \frac{1}{n}$ is not an upper bound of S, but the number $u + \frac{1}{n}$ is.
7. If $S = \left\{\frac{1}{n} - \frac{1}{m} \mid m, n \in \mathbb{N}\right\}$, find $\inf S$ and $\sup S$.

8. Let X be a non-empty set and let $f : X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. If $a \in \mathbb{R}$, show that

$$\sup\{a + f(x) : x \in X\} = a + \sup\{f(x) : x \in X\}$$
$$\inf\{a + f(x) : x \in X\} = a + \inf\{f(x) : x \in X\}.$$

9. Let A and B be bounded non-empty subsets of $\mathbb{R}$, and let

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Prove that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$.

10. Let X be a non-empty set and let $f, g : X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Show that

$$\sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$$
$$\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x) + g(x) \mid x \in X\}.$$

11. Let X and Y be non-empty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $F : X \to \mathbb{R}$ and $G : Y \to \mathbb{R}$ be defined by

$$F(x) = \sup\{h(x,y) \mid y \in Y\} \quad \text{and} \quad G(y) = \sup\{h(x,y) \mid x \in X\}.$$

Show that

$$\sup\{h(x,y) \mid (x,y) \in X \times Y\} = \sup\{F(x) \mid x \in X\} = \sup\{G(y) \mid y \in Y\}.$$

12. Show there exists a positive real number u such that $u^2 = 3$.
13. Show there exists a positive real number u such that $u^3 = 2$.
14. Let $S \subseteq \mathbb{R}$ and suppose that $s^* = \sup S$ belongs to S. If $u \notin S$, show that $\sup(S \cup \{u\}) = \sup\{s^*, u\}$.
15. Show that a non-empty finite set $S \subseteq \mathbb{R}$ contains its supremum.
16. If $S \subseteq \mathbb{R}$ is a non-empty bounded set and $I_S = [\inf S, \sup S]$, show that $S \subseteq I_S$. Moreover, if J is any closed bounded interval of $\mathbb{R}$ such that $S \subseteq J$, show that $I_S \subseteq J$.

17. Prove that if $K_n = (n, \infty)$ for $n \in \mathbb{N}$, then $\displaystyle\bigcap_{n\in\mathbb{N}} K_n = \varnothing$.
18. If S is finite and $s^* \notin S$, show $S \cup \{s^*\}$ is finite.
19. Show directly that there exists a bijection between $\mathbb{Z}$ and $\mathbb{Q}$.
20. Using only the field axioms of $\mathbb{R}$, show that the multiplicative identity of $\mathbb{R}$ is unique.
21. Using only the field axioms of $\mathbb{R}$, show that $(2x-1)(2x+1) = 4x^2 - 1$.
22. Using only the order axioms, usual arithmetic manipulations, and inequalities between concrete numbers, prove that if $x \in \mathbb{R}$ satisfies $x < \varepsilon$ for all $\varepsilon > 0$, then $x \leq 0$.
23. Show that there exists some $x \in \mathbb{R}$ satisfying $x^2 + x = 5$.
24. Consider a set S with $0 \leq \sup S = A < \infty$ and $A \notin S$. Show that for all $\varepsilon > 0$, $S \cap [A-\varepsilon, A] \neq \varnothing$. Using this fact, conclude that $S \cap [A-\varepsilon, A]$ is infinite.
25. Somebody walks up to you with a proof by induction of the statement "*For any integer $N \in \mathbb{N}$, all collections of N sheep are the same colour*," as follows:
    - **Notation:** Let $x_1, x_2, \ldots,$ be the colours of all sheep in the world, in some order.
    - **Base Case:** Obviously the first sheep is a single colour, x_1.

- **Induction Step:** Assume that the statement is true up to some integer n.

  By the induction hypothesis, the collection of the first n sheep $\{x_1, \ldots, x_n\}$ are one colour (label this "colour 1'), and the collection of the last n sheep $\{x_2, \ldots, x_{n+1}\}$ are also one colour (label this "colour 2" - note that we haven't yet shown it is the same colour as the first collection).

  Since $\{x_2, \ldots, x_n\}$ are in both sets, we must have that "colour 1" and "colour 2" are the same, and so $\{x_1, \ldots, x_{n+1}\}$ are all one colour.

Explain why this "proof" fails by identifying/explaining a (significant) false statement.

Solutions

1. **Proof.**

2. **Proof.**

3. **Proof.**

4. **Proof.**

5. **Proof.**

6. **Proof.**

7. **Proof.**

8. **Proof.**

9. **Proof.**

10. **Proof.**

11. **Proof.**

12. **Proof.**

13. **Proof.**

14. **Proof.**

15. **Proof.**

16. **Proof.**

17. **Proof.**

18. **Proof.**

19. **Proof.**

■

20. **Proof.**

∎

21. **Proof.**

∎

22. **Proof.**

$\blacksquare$

23. **Proof.**

24. **Proof.**

$\blacksquare$

25. **Solution.**

■