Mathematical Analysis

Chapter 4
Limits and Continuity

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Overview

The main objects of study in analysis are functions. In this chapter, we

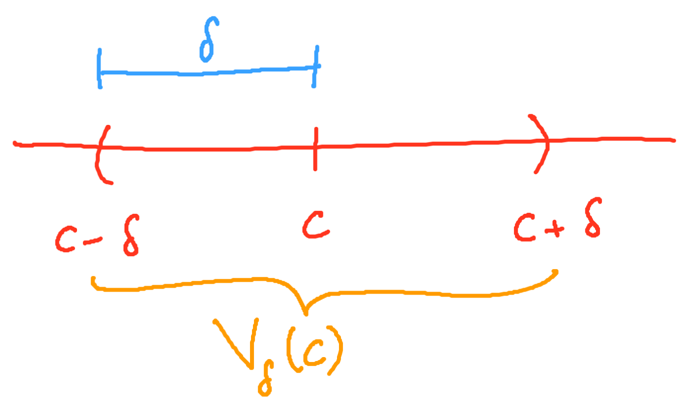
- introduce the $\varepsilon - \delta$ definition of the limit of a function,
- provide results that help to compute such limits,
- identify two types of continuity, and
- present some of the heavy-hitting theorems that form the basis of analytical endeavours.

Outline

4.1 – Limit of a Function

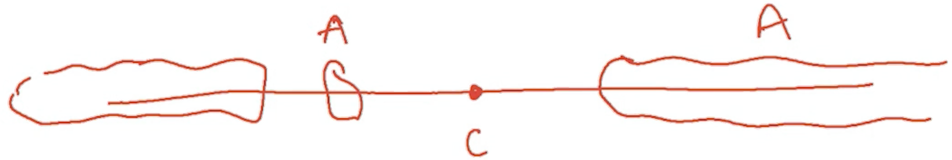
The objects we have studied thus far are functions of $\mathbb{N}$ into $\mathbb{R}$. However, most of calculus deals with functions of $\mathbb{R}$ into $\mathbb{R}$. How do we generalize the concepts and results we have derived for sequences to functions?

Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. The **neighbourhood** $V_\delta(c)$, where $\delta > 0$, is the interval $\{x \in \mathbb{R} \mid |x - c| < \delta\} = (c - \delta, c + \delta)$.

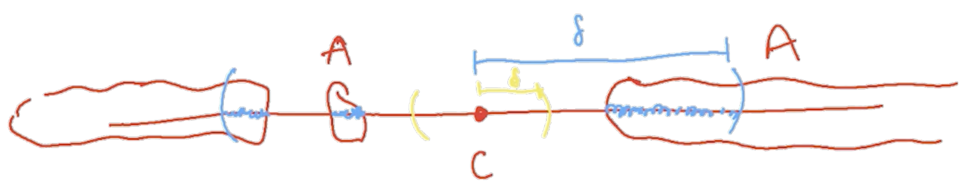


The point $c \in \mathbb{R}$ is a **limit point** (or **cluster point**) of A if every neighbourhood $V_\delta(c)$ contains at least one point $x \in A$ other than c.

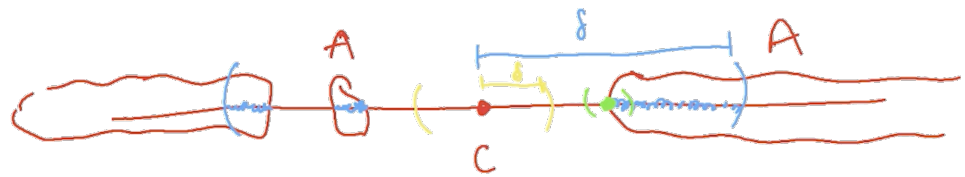
Consider the set $A \subseteq \mathbb{R}$ drawn below.



The $V_\delta(c)-$neighbourhood in blue contains points in A other than c, but c is not a limit point of A since the $V_\delta(c)-$neighbourhood in yellow does not contain points of A.



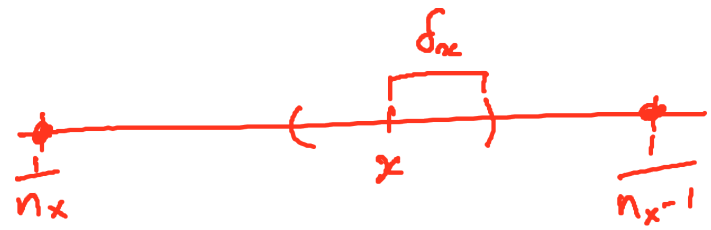
The point at the center of the green interval is a limit point of A, however.



The set of all limit points of A is denoted by $\overline{A}$; a limit point of A does not have to be in A.

Example: What are the limit points of $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$?

Solution.



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Directly determining the limit points of a set is a time-intensive endeavour. Thankfully, there is a link between limit points and convergent sequences.

Theorem 24. *A point $c \in \mathbb{R}$ is a limit point of A if and only if there is a sequence $(a_n) \subseteq A$, with $a_n \neq c$ for $n \in \mathbb{N}$, such that $a_n \to c$.*

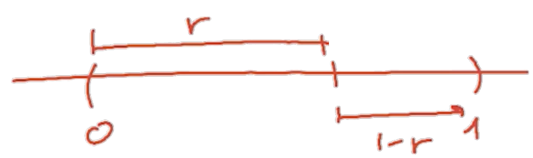
Proof.

■

Any limit point of A is in fact the limit of a sequence in A, and *vice-versa*.

Example: Let $A = [0, 1] \cap \mathbb{Q}$. What are the limit points of A?

Solution.



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Intuitively, a limit of a function f at c is a value L towards which $f(x)$ "approaches" as x gets closer to c, if it exists. But what does that actually mean? What would need to happen for the value not to exist?

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $c \in \overline{A}$: $L \in \mathbb{R}$ is the **limit of f at c** if

$\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0$ such that $0 < |x-c| < \delta_\varepsilon$ and $x \in A \implies |f(x)-L| < \varepsilon.$

This situation is denoted by $\lim_{x \to c} f(x) = L$, or by $f(x) \to L$ when $x \to c$.

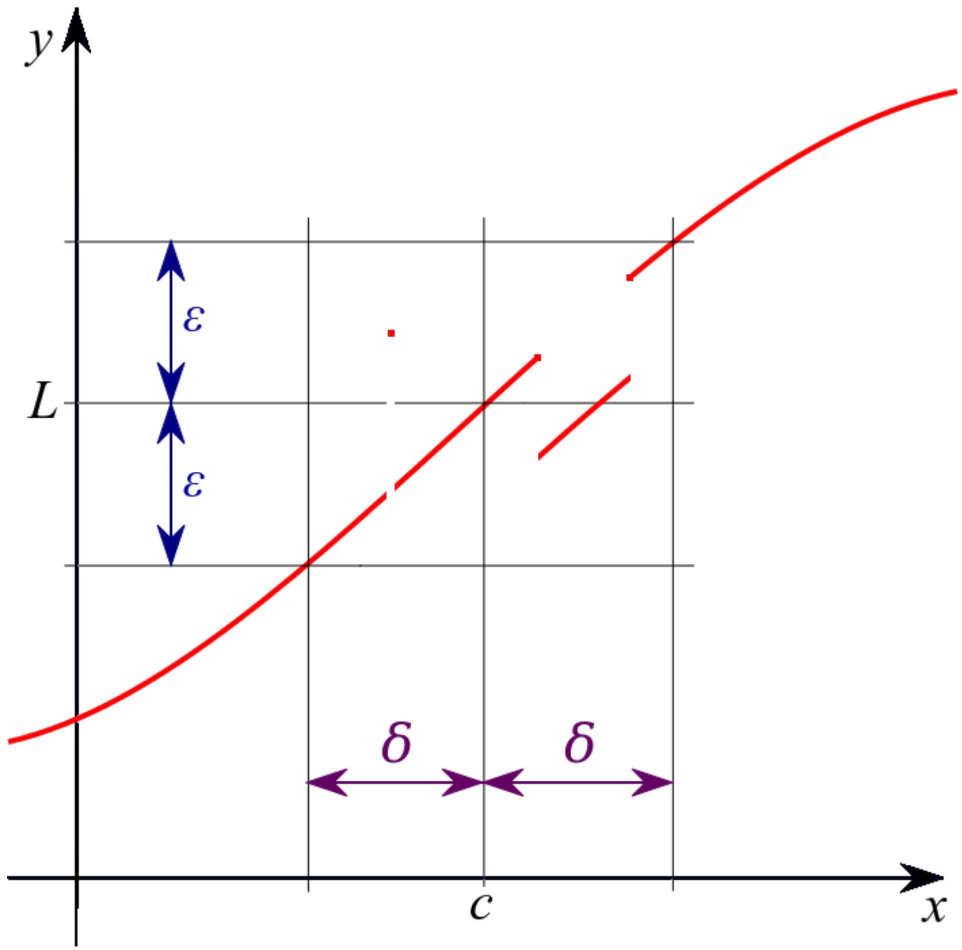
The limit of f at c is **not** $L \in \mathbb{R}$ if

$$\exists \varepsilon_0 > 0,\ \forall \delta > 0,\ \exists x_\delta \in A \text{ such that } 0 < |x_\delta - c| < \delta_\varepsilon \text{ and } |f(x_\delta) - L| \geq \varepsilon_0,$$

which we denote by $\lim_{x \to c} f(x) \neq L$, or by $f(x) \not\to L$ when $x \to c$.

It is the same principle as that of the limit of a sequence: given $\varepsilon > 0$, we need to find a $\delta_\varepsilon > 0$ which satisfies the definition.

Graphically, this is equivalent to putting a horizontal strip of width 2ε around the line $y = L$, and showing that there is a neighbourhood $V_{\delta_\varepsilon}(c)$ such that $f(x)$ is in the strip for any $x \in V_{\delta_\varepsilon}$.



y
L
ε
ε
δ
δ
c
x

Examples:

1. Let $f:[0,1)\to\mathbb{R}$ be the function defined by

$$f(x)=\begin{cases}2, & x\in(0,1)\\ 3, & x=0\end{cases}$$

Show $\lim\limits_{x\to 0} f(x)=2$.

Proof.

$\blacksquare$

2. Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{x^2+2x+2}{x+1}$. Show $\lim_{x \to 2} f(x) = \frac{10}{3}$.

Proof.

$\blacksquare$

3. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^2 \cos(1/x)$. Show that $\lim_{x \to 0} f(x) = 0$.

Proof.

■

As is the case with sequences, a function has at most one limit at any of its limit points c.

Theorem 25. *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and c a limit point of A. Then f has at most one limit at c.*

Proof.

■

As is the case with sequences, the definition is useless if we do not have a candidate for L beforehand. The next result allows us to get such a candidate before using the definition (if required).

Theorem 26. (Sequential Criterion) *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and c a limit point of A. Then*

$$\lim_{x \to c} f(x) = L \quad \textit{if and only if} \quad \lim_{n \to \infty} f(x_n) = L$$

for any sequence $(x_n) \subseteq A$, with $x_n \neq c$ for all $n \in \mathbb{N}$, such that $x_n \to c$.

Proof.

$\blacksquare$

Examples:

1. Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=3x^3+x+1$. Compute $\lim\limits_{x\to 7} f(x)$.

   Solution.

   $\blacksquare$

2. Let $f : (2, \infty) \to \mathbb{R}$, $f(x) = \frac{(x-1)(x-2)}{(x-2)}$. Compute $\lim_{x\to 2} f(x)$.

Solution.

$\blacksquare$

3. Let $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$, $f(x)=x^2\cos(1/x)$. Show that $\lim_{x\to 0} f(x)=0$.

Proof.

$\blacksquare$

4. Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by

$$f(x)=\begin{cases}0, & x\in\mathbb{Q}\\ 1, & x\notin\mathbb{Q}\end{cases}$$

Show that $\lim_{x\to 0} f(x)$ does not exist.

Proof.

$\blacksquare$

5. Let $\mathrm{sgn} : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Show that $\lim\limits_{x \to 0} (x + \mathrm{sgn}(x))$ does not exist.

Proof.

$\blacksquare$

To show that the limit does not exist, it is enough to show that two specific sequences $(x_n), (y_n) \subseteq A$, with $x_n, y_n \neq c$ for all $n \in \mathbb{N}$ and $x_n, y_n \to c$, exist such that $f(x_n) \to L_1$, $f(y_n) \to L_2$, $L_1 \neq L_2$.

But it is not sufficient to find two sequences $(x_n), (y_n) \subseteq A$ with $x_n, y_n \neq c$ for all $n \in \mathbb{N}$, $x_n, y_n \to c$, and $f(x_n), f(y_n) \to L$ to show that the limit exist.

Note that at no point have needed to use the graph of the functions.

4.2 – Properties of Limits

Theorem 27. (Operations on Limits)
Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, and c a limit point of A. Suppose $f(x) \to L$ and $g(x) \to M$ when $x \to c$. Then

1. $\lim\limits_{x\to c} |f(x)| = |L|$;
2. $\lim\limits_{x\to c} (f(x) + g(x)) = L + M$;
3. $\lim\limits_{x\to c} f(x)g(x) = LM$;
4. $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, *if* $g(x) \neq 0$ *for all* $x \in A$ *and if* $M \neq 0$.

Proof.

1.

2.

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4.

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There is also a Squeeze Theorem for functions, but it is not nearly as useful as the corresponding result for sequences.

Theorem 28. (Squeeze Theorem for Functions)
Let $A \subseteq \mathbb{R}$, $f, g, h : A \to \mathbb{R}$, and c a limit point of A. If $f(x) \leq g(x) \leq h(x)$ for all $x \in A$ and if $f(x), h(x) \to L$ when $x \to c$, then $g(x) \to L$ when $x \to c$.

Proof.

■

Examples:

1. Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=k$, $k\in\mathbb{R}$. Show that $\lim\limits_{x\to c} f(x)=k$ for all $c\in\mathbb{R}$.

   Proof.

   $\blacksquare$

2. Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x$. Show that $\lim\limits_{x\to c} f(x)=f(c)$ for all $c\in\mathbb{R}$.

   Proof.

   $\blacksquare$

3. Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\frac{x^3+2x-4}{x^2+1}$. Compute $\lim\limits_{x\to 3} f(x)$.

Solution.

$\blacksquare$

4. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^2 \cos(1/x)$. Show that $\lim_{x \to 0} f(x) = 0$.

Proof.

■

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in \overline{A}$. The function f is **bounded on some neighbourhood of** c if $\exists \delta > 0$ and $M > 0$ are such that $|f(x)| \leq M$ for all $x \in A \cap V_\delta(c)$.

Theorem 29. *If $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in \overline{A}$, and $\lim\limits_{x \to c} f(x) = L$ for some $L \in \mathbb{R}$, then f is bounded on some neighbourhood of c.*

Proof.

$\blacksquare$

4.3 – Continuous Functions

Functions like polynomials, or trigonometric functions, are continuous.

Intuitively, a function is continuous at a point if the graph of the function at that point can be traced without lifting the pen. The notion of "continuity" is fundamental is calculus.

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $c \in A$; f **is continuous at** c if

$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ such that $|x - c| < \delta_\varepsilon$ and $x \in A \implies |f(x) - f(c)| < \varepsilon$.

When computing the limit of f at c, we are interested in the behaviour of the function near c, but not at c. When we are dealing with continuity, we also include the behaviour at c.

When c is a limit point of A, this definition actually means that

$$\lim_{x\to c} f(x) = f(c).$$

If $c \notin \overline{A}$, the expression $\lim_{x\to c} f(c)$ is meaningless since no sequence $(x_n) \subseteq A$ with $x_n \neq c$ for all $n \in \mathbb{N}$ converges to c.

In that case, f is automatically continuous at c. Indeed, there will then be a $\delta > 0$ such that $V_\delta(c)$ contains no point of A but c.

Then for $\varepsilon > 0$, whenever $x \in A$ and $|x - c| < \delta$ (i.e. whenever $x = c$), we have

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon.$$

The definition contains 3 statements: a function f is continuous at c if

1. $f(c)$ is defined;

2. $\lim_{x \to c} f(x)$ exists, and

3. $\lim_{x \to c} f(x) = f(c)$.

Let $B \subseteq A$. If f is continuous for all $c \in B$, then f **is continuous on** B.

Examples:

- Let $f : [0, \infty) \to \mathbb{R}$, $f(x) = \frac{x^2+2x+2}{x+1}$. Is f continuous at $c = 2$?

Solution.

∎

- Let $f:[0,1)\to\mathbb{R}$,

$$f(x)=\begin{cases}2, & x\in(0,1)\\ 3, & x=0\end{cases}$$

Is f continuous at $c=0$?

Solution.

$\blacksquare$

- Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x^3 + x + 1$. Is f continuous at $c = 7$?

Solution.

$\blacksquare$

- Let $f:(2,\infty)\to\mathbb{R}$, $f(x)=\frac{(x-1)(x-2)}{(x-2)}$. Is f continuous at $c=2$?

  Solution.

  ■

- Let $f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases}0, & x\in\mathbb{Q}\\ 1, & x\notin\mathbb{Q}\end{cases}$$

  Is f continuous at $c=0$?

  Solution.

  ■

- Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=k$, $k\in\mathbb{R}$. Is f continuous on $\mathbb{R}$?

  Solution.

  ■

- Let $f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases} x, & x\in\mathbb{Q} \\ 0, & x\notin\mathbb{Q}\end{cases}$$

  Is f continuous at $c=0$?

Solution.

$\blacksquare$

- Same function, but at $c \neq 0$?

Solution.

$\blacksquare$

- Let $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$. Is f continuous on $[0, \infty)$?

  Solution.

$\blacksquare$

- Let $A = \{x \in \mathbb{R} \mid x > 0\}$. Consider the function $f : A \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q}, \text{ with } \gcd(m,n) = 1 \end{cases}$$

Where is f is continuous?

Solution.

Continuity behaves very nicely with respect to elementary operations on functions.

Theorem 30. (Operations on Continuous Functions)
Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, and $c \in A$. If f, g are continuous at c, then

1. *$|f|$ is continuous at c;*
2. *$f + g$ is continuous at c;*
3. *fg is continuous at c;*
4. *$\frac{f}{g}$ is continuous at c if $g \neq 0$ on A.*

Proof.

$\blacksquare$

Corollary. The same results holds if we replace "continuous at c" with "continuous at A".

Since constants and the identity function are continuous on $\mathbb{R}$ (see preceding examples), so are polynomial functions. Furthermore, rational functions are continuous on their domain.

The **composition** of the functions $f : A \to B$ and $g : B \to C$ is the function $g \circ f : A \to C$, with $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

Theorem 31. (Composition of Continuous Functions)
Let $A, B \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, $c \in A$. If f is continuous at c, g is continuous at $f(c)$, and $f(A) \subseteq B$, then $g \circ f : A \to B$ is continuous at c.

Proof.

$\blacksquare$

Corollary. The same results holds if we replace "continuous at c" with "continuous at A".

Example: Let $f : [0,\infty) \to \mathbb{R}$, $f(x) = \sqrt{3x^3 + x + 1}$. Show that f is continuous on $[0,\infty)$.

Proof.

$\blacksquare$

An **algebraic** function is a function obtained via the (possibly repeated) composition of rational functions and root functions. The class of algebraic functions is continuous on it domain. The same goes for trigonometric, exponential, and logarithmic functions, *via* their power series definition.

4.4 – Max/Min Theorem

We now begin our study of the classical theorems of calculus.

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$. The function $f : A \to \mathbb{R}$ is **bounded** on A if $\exists M > 0$ such that $|f(x)| < M$ for all $x \in A$.

Examples:

1.

2.

3.

Theorem 32. *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f is bounded on $[a, b]$.*

Proof.

■

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$. We say that f **reaches a global maximum on** A if $\exists x^* \in A$ such that $f(x^*) \geq f(x)$ for all $x \in A$.

Similarly, f **reaches a global minimum on** A if $\exists x_* \in A$ such that $f(x_*) \leq f(x)$ for all $x \in A$.

Continuous functions on closed, bounded sets have a useful property.

Theorem 33. (Max/Min Theorem)
If $f : [a, b] \to \mathbb{R}$ is continuous, then f reaches a global maximum and a global minimum of $[a, b]$.

Proof.

Examples:

1.

2.

⚠ The hypotheses of a theorem have to be satisfied in order to use it to draw its conclusion, but the conclusion is not necessarily false if the hypotheses are not met.

3.

4.

4.5 – Intermediate Value Theorem

The following result has many implications; it can notably be used to locate the roots of a function.

Theorem 34. *Let $f:[a,b]\to\mathbb{R}$ be continuous. If $\exists\alpha,\beta\in[a,b]$ such that $f(\alpha)f(\beta)<0$, then $\exists\gamma\in(a,b)$ such that $f(\gamma)=0$.*

Proof.

∎

Example: Show that $\exists x \in \mathbb{R}^+$ such that $x^2 = 2$.

Proof.

∎

This result easily generalizes to the following.

Theorem 35. (Intermediate Value Theorem)
Let $f : [a, b] \to \mathbb{R}$ *be continuous. If* $\exists \alpha < \beta \in [a, b]$ *s.t.* $f(\alpha) < k < f(\beta)$ *or* $f(\alpha) > k > f(\beta)$*, then* $\exists \gamma \in (a, b)$ *such that* $f(\gamma) = k$*.*

Proof.

$\blacksquare$

The following result combines the Max/Min Theorem and the Intermediate Value Theorem.

Theorem 36. *If $f : [a,b] \to \mathbb{R}$ is continuous, then $f([a,b])$ is a closed and bounded interval.*

Proof.

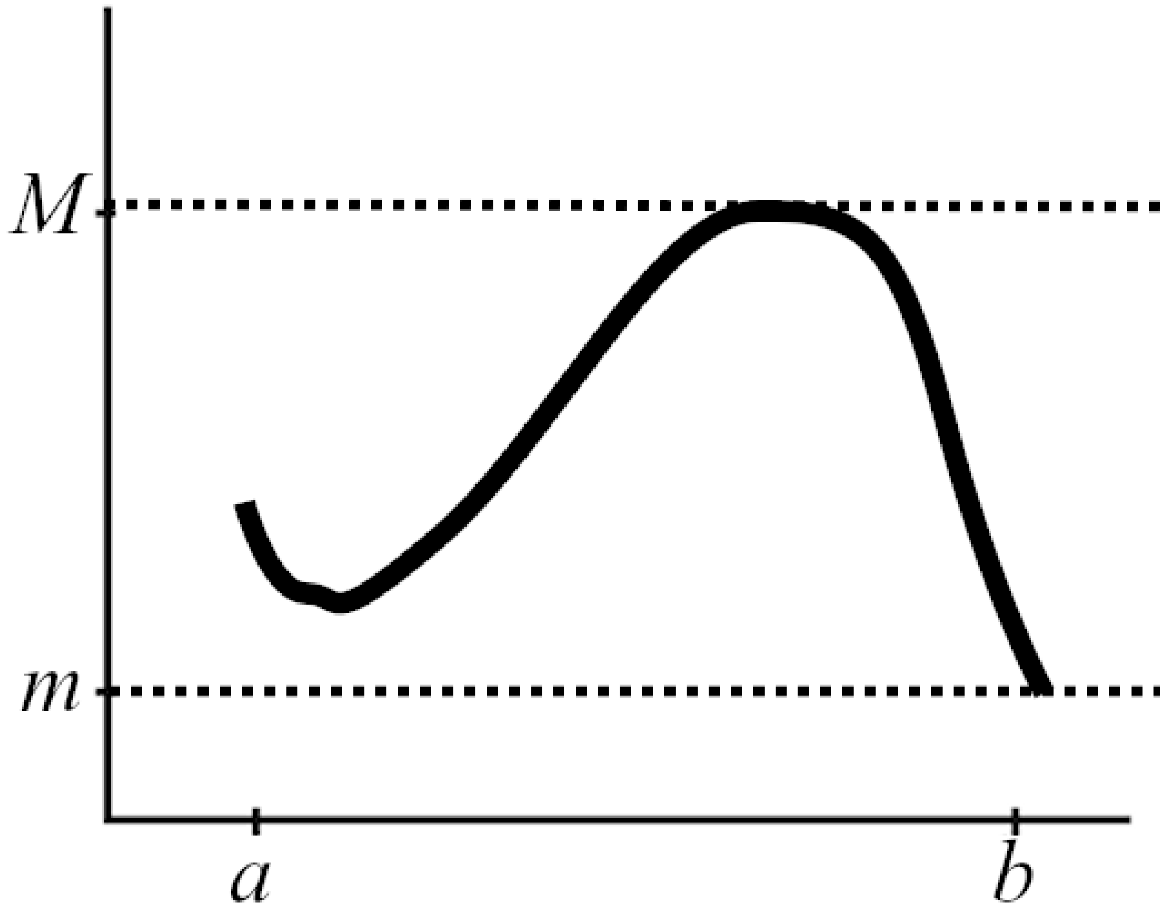
$\blacksquare$

The image of any interval by a continuous function is always an interval, but the only time that we know for a fact that image is of the same type as the original is when the original is closed and bounded.

Examples:

1.

2.


M
m
a
b

4.6 – Uniform Continuity

Let $f : A \to \mathbb{R}$ be continuous (on A). For $\varepsilon > 0$ and $c \in A$, the $\delta_\varepsilon > 0$ that is used to show continuity of f at c depends generally on ε **and** c. But there are instances when δ_ε depends only on ε.

The function f is **uniformly continuous** on A if

$$x, y \in A \text{ and } |x - y| < \delta_\varepsilon \implies |f(x) - f(y)| < \varepsilon.$$

The notion of uniform continuity is more restrictive than that of (simple) continuity.

Theorem 37. *If $f : A \to \mathbb{R}$ is uniformly continuous on A, then f is continuous on A.*

Proof.

$\blacksquare$

Example: Show that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous on $(0, \infty)$ but not uniformly continuous on $(0, \infty)$.

Proof.

■

Lemma. If f is uniformly continuous on A and $(x_n) \subseteq A$ is a Cauchy sequence, then $f(x_n)$ is a Cauchy sequence.

Proof.

■

While continuous functions are not generally uniformly continuous, there is a specific class of functions for which continuity is equivalent to uniform continuity.

Theorem 38. *Let $f : [a, b] \to \mathbb{R}$. Then f is uniformly continuous on $[a, b]$ if and only if f is continuous on $[a, b]$.*

Proof.

$\blacksquare$

Example: Show $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\frac{1}{1+x^2}$ is uniformly continuous on $[0,1]$.

Proof.

$\blacksquare$

Note that $\forall z\in\mathbb{R}$, $0\leq (|z|-1)^2=z^2-2|z|+1 \implies 2|z|\leq 1+z^2$.

4.7 – Exercises

1. Let $f : \mathbb{R} \to \mathbb{R}$ and let $c \in \mathbb{R}$. Show that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to 0} f(x+c) = L$.
2. Show $\lim_{x\to c} x^3 = c^3$ for any $c \in \mathbb{R}$.
3. Use either the $\varepsilon - \delta$ definition of the limit or the Sequential Criterion for limits to establish the following limits:
   (a) $\displaystyle\lim_{x\to 2} \frac{1}{1-x} = -1$;
   (b) $\displaystyle\lim_{x\to 1} \frac{x}{1+x} = \frac{1}{2}$;
   (c) $\displaystyle\lim_{x\to 0} \frac{x^2}{|x|} = 0$, and
   (d) $\displaystyle\lim_{x\to 1} \frac{x^2-x+1}{x+1} = \frac{1}{2}$

4. Show that the following limits do not exist:
   (a) $\lim_{x\to 0} \frac{1}{x^2}$, with $x > 0$;
   (b) $\lim_{x\to 0} \frac{1}{\sqrt{x}}$, with $x > 0$;
   (c) $\lim_{x\to 0}(x + \operatorname{sgn}(x))$, and
   (d) $\lim_{x\to 0} \sin(1/x^2)$, with $x > 0$.
5. Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to c} (f(x))^2 = L$. Show that if $L = 0$, then $\lim_{x\to c} f(x) = 0$. Show that if $L \neq 0$, then f may not have a limit at c.
6. Let $f : \mathbb{R} \to \mathbb{R}$, let J be a closed interval in $\mathbb{R}$ and let $c \in J$. If f_2 is the restriction of f to J, show that if f has a limit at c then f_2 has a limit at c. Show the converse is not necessarily true.

7. Determine the following limits and state which theorems are used in each case.
   (a) $\lim_{x\to 2} \sqrt{\dfrac{2x+1}{x+3}}$, $(x > 0)$;
   (b) $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$, $(x > 0)$;
   (c) $\lim_{x\to 0} \sqrt{\dfrac{(x+1)^2-1}{x}}$, $(x > 0)$, and
   (d) $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$, $(x > 0)$.
8. Give examples of functions f and g such that f and g do not have limits at point c, but both $f+g$ and fg have limits at c.

9. Determine whether the following limits exist in $\mathbb{R}$:
   (a) $\lim\limits_{x\to 0} \sin\left(\dfrac{1}{x^2}\right)$, with $x \neq 0$;
   (b) $\lim\limits_{x\to 0} x\sin\left(\dfrac{1}{x^2}\right)$, with $x \neq 0$;
   (c) $\lim\limits_{x\to 0} \operatorname{sgn}\sin\left(\dfrac{1}{x}\right)$, with $x \neq 0$, and
   (d) $\lim\limits_{x\to 0} \sqrt{x}\sin\left(\dfrac{1}{x^2}\right)$, with $x > 0$.
10. Let $f : \mathbb{R} \to \mathbb{R}$ be s.t. $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Assume $\lim\limits_{x\to 0} f(x) = L$ exists. Prove that $L = 0$ and that f has a limit at every point $c \in \mathbb{R}$.
11. Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition

$$|f(x) - f(y)| \leq K|x-y|$$

for all $x, y \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$.

12. Let $f : (0,1) \to \mathbb{R}$ be bounded and s.t. $\lim_{x\to 0} f(x)$ does not exist. Show that there are two convergent sequences $(x_n), (y_n) \subseteq (0,1)$ with $x_n, y_n \to 0$ and $f(x_n) \to \xi$, $f(y_n) \to \zeta$, but $\xi \neq \zeta$.
13. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $P = \{x \in \mathbb{R} : f(x) > 0\}$. If $c \in P$, show that there exists a neighbourhood $V_\delta(c) \subseteq P$.
14. Prove that if an additive function is continuous at some point $c \in \mathbb{R}$, it is continuous on $\mathbb{R}$.
15. If f is a continuous additive function on $\mathbb{R}$, show that $f(x) = cx$ for all $x \in \mathbb{R}$, where $c = f(1)$.
16. Let $I = [a,b]$ and $f : I \to \mathbb{R}$ be a continuous function on I s.t. $\forall x \in I$, $\exists y \in I$ s.t. $|f(y)| \leq \frac{1}{2}|f(x)|$. Show $\exists c \in I$ s.t. $f(c) = 0$.
17. Show that every polynomial with odd degree has at least one real root.
18. Let $f : [0,1] \to \mathbb{R}$ be continuous and s.t. $f(0) = f(1)$. Show $\exists c \in [0, \frac{1}{2}]$ s.t. $f(c) = f(c + \frac{1}{2})$.

19. Determine whether the following limits exist in $\mathbb{R}$:
    (a) $\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$, with $x \neq 0$;
    (b) $\lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right)$, with $x \neq 0$;
    (c) $\lim_{x\to 0} \operatorname{sgn}\sin\left(\frac{1}{x}\right)$, with $x \neq 0$, and
    (d) $\lim_{x\to 0} \sqrt{x}\sin\left(\frac{1}{x^2}\right)$, with $x > 0$.
20. If $f(x) = x$ and $g(x) = \sin x$, show that f and g are both uniformly continuous on $\mathbb{R}$ but that their product is not uniformly continuous on $\mathbb{R}$.
21. Let $A \subseteq \mathbb{R}$ and suppose that f has the following property: $\forall \varepsilon > 0$, $\exists g_\varepsilon : A \to \mathbb{R}$ s.t. g_ε is uniformly continuous on A with $|f(x) - g_\varepsilon(x)| < \varepsilon$ for all $x \in A$. Show f is uniformly continuous on A.
22. Prove that a continuous $p-$periodic fonction on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

23. Define $g:\mathbb{R}\to\mathbb{R}$ by

$$g(x)=\begin{cases}\frac{(-1)^n}{n} & \text{if } x=1/n \text{ for some } n\in\mathbb{N},\\ 0 & \text{otherwise}\end{cases}$$

Prove that g is continuous at 0.

24. Let $f:\mathbb{R}\to\mathbb{R}$. The **pre-image** of a subset $B\subseteq\mathbb{R}$ under f is

$$f^{-1}(B)=\{a\in A \mid f(a)\in B\}.$$

Prove that f is continuous if and only if the pre-image of every open subset of $\mathbb{R}$ is an open subset of $\mathbb{R}$.

25. A function $f:A\to\mathbb{R}$ is said to be **Lipschitz** if there is a positive number M such that

$$|f(x)-f(y)|\leq M|x-y| \quad \forall\, x,y\in A.$$

Show that a Lipschitz function must be uniformly continuous, but that uniformly continuous functions do not have to be Lipschitz.

Solutions

1. **Proof.**

2. **Proof.**

3. **Proof.**

4. **Proof.**

5. **Proof.**

6. **Proof.**

7. **Proof.**

8. **Proof.**

9. **Proof.**

10. **Proof.**

11. **Proof.**

12. **Proof.**

13. **Proof.**

14. **Proof.**

15. **Proof.**

16. **Proof.**

17. **Proof.**

18. **Proof.**

19. **Proof.**

20. **Proof.**

21. **Proof.**

22. **Proof.**

23. **Proof.**

24. **Proof.**

25. **Proof.**