Mathematical Analysis

Chapter 7
Series of Functions

P. Boily (uOttawa)

Winter 2022

Overview

We discuss a specific type of sequence: the series.

In particular, we will discuss

- series of numbers,
- series of functions, and
- power series.

The latter is more naturally expressed using a complex analysis framework, but we will present it, and important theorems for regular series, in the real analysis framework.

Outline

7.1 – Series of Numbers (p.3)

7.2 – Series of Functions (p.21)

7.3 – Power Series (p.29)

7.4 – Exercises (p.44)

7.1 – Series of Numbers

Let $(x_n) \subseteq \mathbb{R}$. The **series** associated with (x_n), denoted by

$$S_{(x_n)} = \sum_{n=1}^{\infty} x_n,$$

is the sequence (s_n), where

$$\begin{aligned} s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ s_3 &= x_1 + x_2 + x_3 \\ &\cdots \end{aligned}$$

If the sequence of partial sums s_n converges to S, we say the series $S_{(x_n)}$ **converges to the sum** S.

We start by producing a necessary condition for convergence.

Theorem 70. *If* $\sum_{n=1}^{\infty} x_n$ *converges, then* $x_n \to 0$.

Proof.

■

We can bypass the need to know the limit in order to prove convergence.

Theorem 71. (Cauchy Criterion for Series)
The series $\sum_{n=1}^{\infty} x_n$ *converges if and only if* $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ *such that*

$$m > n > N_\varepsilon \implies |x_{n+1} + \cdots + x_m| < \varepsilon.$$

Proof.

■

But there are other tests that can be used to show the convergence of a series without knowing the limit.

Theorem 72. (Comparison Test)
Let $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ *be series whose terms are all non-negative. If* $\exists K \in \mathbb{N}$ *such that* $0 \leq x_n \leq y_n$ *when* $n > K$, *then*

1. $\sum_{n=1}^{\infty} y_n$ *converges* $\implies \sum_{n=1}^{\infty} x_n$ *converges.*

2. $\sum_{n=1}^{\infty} x_n$ *diverges* $\implies \sum_{n=1}^{\infty} y_n$ *diverges.*

Proof.

■

Examples: Discuss the convergence of 1. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ and 2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

1.

■

2.

■

Theorem 73. (Alternating Series Test)
Let (a_n) *be a sequence of non-negative numbers such that* $a_n \searrow 0$ *(i.e* $a_n \to 0$ *and* $a_{n+1} \leq a_n$*). Then* $\sum_{n=0}^{\infty}(-1)^n a_n$ *converges.*

Proof.

■

Even though it was not part of the statement of the Alternating Series Test, the proof allows us to conclude that the value of a convergent alternating series lies between a_{2k} and a_{2m+1} for all $k, m \in \mathbb{N}$.

Example: The alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Proof.

■

Two other convergence tests are often used in practice: the Ratio Test and the Root Test. We shall prove only the Ratio Test, the proof for the Root Test is similar.

Theorem 74. (Ratio Test)
Let (a_n) be a sequence of positive real numbers.

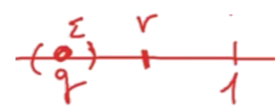
1. *If* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

2. *If* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

If $\frac{a_{n+1}}{a_n} \to 1$, then the series may converge or diverge, depending on the nature of the terms a_n.

Proof.

1.



2.

■

The key parts of the proof (namely, the convergence of the tail in the first case and the condition $a_n \not\to 0$ in the second) are also valid if the statement is relaxed to some extent.

Theorem 74. (Ratio Test – Reprise)
Let (a_n) be a sequence of real numbers with $a_n \neq 0$ for all n.

1. *If* $\limsup_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

2. *If* $\liminf_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Theorem 75. (Root Test)
Let (a_n) *be a sequence of positive real numbers.*

1. *If* $\limsup_{n\to\infty} \sqrt[n]{a_n} < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

2. *If* $\liminf_{n\to\infty} \sqrt[n]{a_n} > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

If $\sqrt[n]{a_n} \to 1$, then the series may converge or diverge, depending on the nature of the terms a_n.

The proof of the Root Test follows the same general lines.

Examples: Discuss the convergence of $\sum_{n=1}^{\infty}\frac{(-1)^n}{n2^n}$; $\sum_{n=1}^{\infty}\frac{3^n}{n2^n}$; $\sum_{n=1}^{\infty}\frac{1}{n^p}$, $p>0$.

1.

■

2.

■

3.

■

Theorem 76. (Absolute Convergence)
If the series $\sum_{n=0}^{\infty} |a_n|$ *converges, so does* $\sum_{n=0}^{\infty} a_n$ *(not an "iff" statement).*

Theorem 77. (Series Rearrangement)
If the series $\sum_{n=0}^{\infty} |a_n|$ *converges, so does* $\sum_{n=0}^{\infty} a_{\varphi(n)}$*,* $\varphi : \mathbb{N} \to \mathbb{N}$ *a bijection.*

7.2 – Series of Functions

Series of functions play the same role for sequences of functions that series played for sequences of numbers.

Let $I \subseteq \mathbb{R}$ and $f_n : I \to \mathbb{R}$, $\forall n \in \mathbb{N}$. If the sequence of partial sums

$$\begin{aligned} s_1(x) &= f_1(x) \\ s_2(x) &= f_1(x) + f_2(x) \\ &\cdots \end{aligned}$$

converges to some function $f : I \to \mathbb{R}$ for all $x \in I$, we say that the series of functions $\sum_{n=1}^{\infty} f_n$ **converges pointwise to** f **on** I.

Example: Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x^n$ for each $n \in \mathbb{N}$. Does the sequence of partial sums $s_k(x)$ converge to some pointwise limit over some $A \subseteq \mathbb{R}$?

Solution.

■

If the sequence of partial sums (s_n) converges uniformly to f on I, we say that the series of functions $\sum_{n=1}^{\infty} f_n$ **converges uniformly to f on I.**

If the convergence of the series of functions is uniform, the limit interchange theorems can be applied.

Theorem 78. (Cauchy Criterion for Series of Functions)
Let $f_n : I \to \mathbb{R}$ for all $n \in \mathbb{N}$. The series of functions with term f_n converges uniformly to some function $f : I \to \mathbb{R}$ if and only if $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ (independent of $x \in I$) such that

$$m > n > N_\varepsilon \implies \left| \sum_{i=n+1}^{m} f_i(x) \right| < \varepsilon.$$

Proof.

■

The next result is a powerful tool to prove uniform convergence (and so to be able to use the Limit Interchange Theorems).

The simplicity of its proof belies its importance.

Theorem 79. (WEIERSTRASS M−TEST)
Let $f_n : I \to \mathbb{R}$ and $M_n \geq 0$ for all $n \in \mathbb{N}$. Assume that $|f_n(x)| \leq M_n$ for all $x \in I$, $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} M_n \text{ converges} \implies \sum_{n=1}^{\infty} f_n \text{ converges uniformly on } I.$$

Proof.

■

Example: Let $\varepsilon \in (0,1)$. Consider the sequence of functions $g_n : \mathbb{R} \to \mathbb{R}$ defined by $g_n(x) = nx^{n-1}$ for each $n \in \mathbb{N}$. Does $\sigma_k(x) \rightrightarrows \sigma(x)$ on $I_\varepsilon = (-1+\varepsilon, 1-\varepsilon)$ for some σ? If so, find σ.

Solution.

■

Incidentally, Theorem 68 also tells us that $s_k(x) \rightrightarrows \frac{1}{1-x}$ on I_ε, for all $0 < \varepsilon < 1$, and that for all $k \in \mathbb{N}$ and $x \in I_\varepsilon$, $\varepsilon \in (0,1)$, we have

$$\sum_{n=0}^{\infty} \frac{d^k}{dx^k}[x^n] = \frac{d^k}{dx^k} \sum_{n=0}^{\infty} x^n = \frac{d^k}{dx^k}\Big(\frac{1}{1-x}\Big)$$

7.3 – Power Series

A **power series** around its **center** $x = x_0$ is a formal expression of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

We have already seen an example of such a series, which converged uniformly on intervals containing $x_0 = 0$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{on } I_\varepsilon = (-1+\varepsilon, 1-\varepsilon),\ \forall \varepsilon \in (0,1)$$

(note, however, that the convergence is only pointwise on $(-1, 1)$).

Furthermore, the function $f : A \to \mathbb{R}$, $f(x) = \frac{1}{1-x}$ is defined for all $x \neq 1$, yet the power series $1+x+x^2+\cdots$ does not converge to f outside of $(-1, 1)$.

Power series are commonly used as a formal guessing procedure to solve differential equations, but this is not a topic we will tackle at the moment.

A natural question to ask is: for which functions $f : A \to \mathbb{R}$ (and which A) can we find a sequence of coefficients (a_n) such that

$$f(x) = \sum_{n=0}^{\infty} a_n, \quad \forall x \in A?$$

Questions of this ilk are more naturally answered in $\mathbb{C}$; a more complete treatment will be provided in a complex analysis course.

Examples: Where do the following power series converge?

$$1.\ \sum_{n=0}^{\infty} x^n, \quad 2.\ \sum_{n=1}^{\infty} (nx)^n, \quad 3.\ \sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n.$$

1. ■

2. ■

3.

■

The **radius of convergence** of a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.$$

If the limit exists, we can replace $\limsup$ by $\lim$. Intuitively, this says that for all large enough n,

$$-R^{-n} \leq -|a_n| \leq a_n \leq |a_n| \leq R^{-n},$$

so that

$$-\sum_{n>N} \left(\frac{x-x_0}{R}\right)^n \leq \sum_{n>N} a_n(x-x_0)^n \leq \sum_{n>N} \left(\frac{x-x_0}{R}\right)^n.$$

The bounds are geometric series, and they converge when $|x-x_0| < R$.

We would expect the original power series to converge on the **interval of convergence** $|x-x_0| < R$.

Theorem 80. *Let R be the radius of convergence of the power series*

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

Then, if

- $R=0$, *the power series converges for $x=x_0$ and diverges for $x\neq x_0$;*
- $R=\infty$, *the power series converges absolutely on $\mathbb{R}$, and*
- $0<R<\infty$, *the power series converges absolutely on $|x-x_0|<R$, diverges on $|x-x_0|>R$; the extremities must be analyzed separately.*

Proof. ■

Theorem 81. *The power series of Theorem 80 converges uniformly on any compact sub-interval*

$$[a,b] \subseteq (x_0 - R, x_0 + R).$$

Proof.

■

In what follows, we let $f : (x_0 - R, x_0 + R) \to \mathbb{R}$ be the function defined by

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad \text{and} \quad s_N(x) = \sum_{n=0}^{N} a_n (x - x_0)^n.$$

Theorem 82. *The function f is continuous on any closed bounded interval $[a, b] \subseteq (x_0 - R, x_0 + R)$.*

Proof.

$\blacksquare$

Theorem 83. *Let $x \in (x_0 - R, x_0 + R)$. Then f is Riemann-integrable between x_0 and x and*

$$\int_{x_0}^{x} f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}.$$

Proof.

■

Theorem 84. *The function f is differentiable on $(x_0 - R, x_0 + R)$ and*

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}.$$

Proof.

■

How do we compute the power series coefficients a_n? Combining Theorems 82 and 84, we see that f is **smooth** in its interval of convergence (i.e. all of its derivatives are continuous).

Theorem 85. *If $R > 0$, then*

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

Proof.

■

Corollary. If $\exists r > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

and $f(x) = g(x)$ for all $x \in (x_0 - r, x_0 + r)$, then $a_n = b_n$ for all $n \in \mathbb{N}$.

Attempts to strengthen this uniqueness result must fail.

Example: Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \exp(-1/x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that f does not have a power series expansion.

Proof.

■

Thus, we cannot always assume that a function is equal to its power series.

There are other ways to expand a function as infinite series, most notable being **Laurent Series** and **Fourier Series**. These topics are covered in courses in complex analysis and partial differential equations, respectively.

7.4 – Exercises

1. Answer the following questions about series.
   (a) If $\sum_{k=1}^{\infty}(a_k + b_k)$ converges, what about $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$?
   (b) If $\sum_{k=1}^{\infty}(a_k + b_k)$ diverges, what about $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$?
   (c) If $\sum_{k=1}^{\infty}(a_{2k} + a_{2k-1})$ converges, what about $\sum_{k=1}^{\infty} a_k$?
   (d) If $\sum_{k=1}^{\infty} a_k$ converges, what about $\sum_{k=1}^{\infty}(a_{2k} + a_{2k-1})$?
2. Show that
$$\frac{1}{r-1} = \frac{1}{r+1} + \frac{2}{r^2+1} + \frac{4}{r^4+1} + \frac{8}{r^8+1} + \cdots$$
   for all $r > 1$.

3. Using Riemann integration, find the values of p for which the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges (compare with the approach used in the notes).

4. Which of the following series converge?

   (a) $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^2}$

   (b) $\sum_{n=1}^{\infty} \frac{2+\sin^3(n+1)}{2^n+n^2}$

   (c) $\sum_{n=1}^{\infty} \frac{1}{2^n-1+\cos^2 n^3}$

   (d) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$

   (e) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5^n}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$

(i) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n+3n^3}{7n^3+2}\right)^n$

5. Give an example of a power series $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ with interval of convergence $[-\sqrt{2}, \sqrt{2})$.

6. Find the values of x for which the following series converge:

(a) $\displaystyle\sum_{n=1}^{\infty} (nx)^n$;

(b) $\displaystyle\sum_{n=1}^{\infty} x^n$;

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$;

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$.

7. If the power series $\sum a_k x^k$ has radius of convergence R, what is the radius of convergence of the series $\sum a_k x^{2k}$?

8. Obtain power series expansions for the following functions.

(a) $\displaystyle\frac{x}{1+x^2}$;

(b) $\displaystyle\frac{x}{(1+x^2)^2}$;

(c) $\displaystyle\frac{x}{1+x^3}$;

(d) $\displaystyle\frac{x^2}{1+x^3}$;

(e) $\displaystyle f(x) = \int_0^1 \frac{1-e^{-sx}}{s}\, ds$, about $x = 0$.

Solutions

1. **Proof.**

2. **Proof.**

3. **Proof.**

4. **Proof.**

5. **Proof.**

6. **Proof.**

7. **Proof.**

8. **Proof.**