Mathematical Analysis

Chapter 9
Metric Spaces and Sequences

P. Boily (uOttawa)

Winter 2022

Overview

One of the natural ways we can extend the concepts we have discussed in the previous chapters is by moving from $\mathbb{R}$ to $\mathbb{R}^m$.

Some of the notions that generalize nicely to vectors and functions on vectors include norms and distances, sequences, and continuity.

Notation: The symbol $\mathbb{K}$ is sometimes used to denote either $\mathbb{R}$ or $\mathbb{C}$.

$C_{\mathbb{R}}([0,1])$ represents the $\mathbb{R}-$vector space of continuous functions $[0,1] \to \mathbb{R}$.

$\mathcal{F}_{\mathbb{R}}([0,1])$ represents the $\mathbb{R}-$vector space of functions $[0,1] \to \mathbb{R}$.

Outline

9.1 – Preliminaries (p.3)

- Norms and Metrics (p.4)
- Metric Space Topology (p.22)
- Continuity (p.50)

9.2 – Sequence in a Metric Space (p.68)

- Closure, Closed Subsets, and Continuity (p.72)
- Complete Spaces and Cauchy Sequences (p.75)

9.3 – Exercises (p.90)

9.1 – Preliminaries

Most of the results of the previous chapters rely heavily on the properties of the absolute value.

Its fundamental role in $\mathbb{R}$ is being a measure of the **magnitude** of a real number: $|x|$ is the distance from the real number x to the origin.

In higher-dimensional spaces, the concept of the absolute value can be generalized in multiple ways.

In this chapter, we discuss **norms** and **metrics**, as well as the topologies they induce.

9.1.1 – Norms and Metrics

Let E be a $\mathbb{K}-$vector space, such as $\mathbb{R}$, $\mathbb{C}^n$ or $C_{\mathbb{R}}([0,1])$, say. A **norm** over E is a mapping $\|\cdot\|: E \to \mathbb{R}$ for which the following properties hold:

1. $\forall \mathbf{x} \in E$, $\|\mathbf{x}\| \geq 0$
2. $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$
3. $\forall \mathbf{x} \in E, \forall \lambda \in \mathbb{K}$, $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$
4. $\forall \mathbf{x}, \mathbf{y} \in E$, $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

If the 4 properties hold, we say of $(E, \|\cdot\|)$ that it is a **normed space**.

Examples:

1.

2.

3.

4.

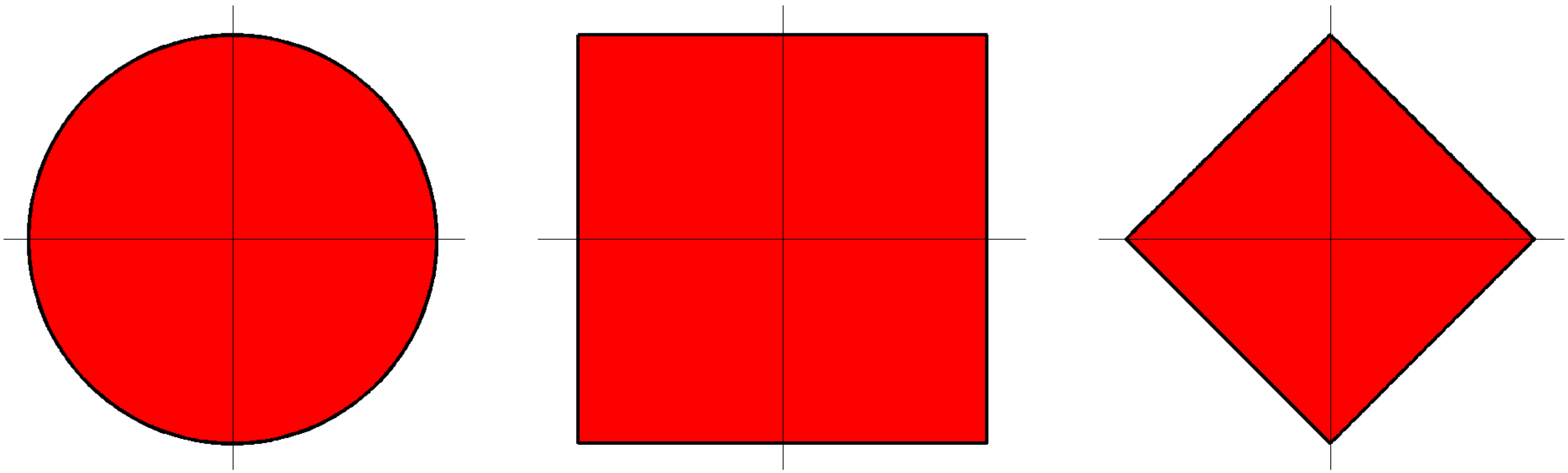
5.

The **open ball of radius** 1 **induced by the** $p-$**norm around the origin in** $\mathbb{R}^n$ is the set of vectors

$$B^p(\mathbf{0},1)=\{\mathbf{x}\mid\|\mathbf{x}\|_p<1\}.$$

There are equivalent definitions for closed balls, or for general balls of radius r centered at some point $\mathbf{a}\in\mathbb{R}^n$.

Different values of p lead to different geometrical sets $B^p(\mathbf{0},1)$.



$B^p(\mathbf{0}, 1)$, for $p = 2, \infty, 1$ (left to right).

The open balls have different shapes, but we shall soon see that they are all equivalent, in the sense that they all induce the same topologies.

There are similarities between summations and integration (the Riemann-integral of a function over an interval is, essentially, the limit of a sum).

As such, it is tempting to conclude that there are equivalent $p-$norms over $\mathcal{F}_{\mathbb{R}}([0,1])$: something along the lines of

$$\|f\|_p = \Big(\int_{[0,1]} |f|^p \, dm \Big)^{1/p} \tag{1}$$

where m is the Lebesgue measure, but these mappings are not norms.

Consider the function $\chi_{\mathbb{Q}} \in \mathcal{F}_{\mathbb{R}}([0,1])$, say. It can be shown that $\|f\|_1 = 0$. However, $\chi_{\mathbb{Q}} \neq 0$ which contradicts the second property of norms (in fact, $\|\cdot\|_p$ is a **seminorm** on $\mathcal{F}_{\mathbb{R}}([0,1])$).

If we restrict the function space to $C_{\mathbb{R}}([0,1])$, $\|\cdot\|_p$ is indeed a norm for all $p \geq 1$, but unfortunately, $(C_{\mathbb{R}}([0,1]), \|\cdot\|_p)$ is not complete (more on this later).

Let E be any set. A **metric** over E is a mapping $d : E \times E \to \mathbb{R}$ for which the following properties hold:

1. $\forall \mathbf{x}, \mathbf{y} \in E$, $d(\mathbf{x}, \mathbf{y}) \geq 0$
2. $\forall \mathbf{x} \in E$, $d(\mathbf{x}, \mathbf{x}) = 0$
3. $d(\mathbf{x}, \mathbf{y}) = 0 \Longleftrightarrow \mathbf{x} = \mathbf{y}$
4. $\forall \mathbf{x}, \mathbf{y} \in E$, $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$
5. $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in E$, $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$

If the 5 properties hold, we say that (E, d) is a **metric space**.

Normed spaces give rise to metric spaces.

Theorem 88. *If* $(E, \|\cdot\|)$ *is a normed space, define* $d : E \times E \to \mathbb{R}$ *by*

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

Then (E, d) *is a metric space.*

Proof.

■

Not every metric space arises from a norm, however.

Examples:

1.

2.

Let (E, d) be a metric space. The **open ball centered at** $\mathbf{a} \in E$ **with radius** $r > 0$ is the set

$$B(\mathbf{a}, r) = B + d(\mathbf{a}, r) = \{\mathbf{x} \in E \mid d(\mathbf{a}, \mathbf{x}) < r\};$$

the **closed ball centered at** $\mathbf{a}$ **with radius** $r > 0$ is the set

$$D(\mathbf{a}, r) = D_d(\mathbf{a}, r) = \{\mathbf{x} \in E \mid d(\mathbf{a}, \mathbf{x}) \leq r\},$$

and the **sphere centered at** $\mathbf{a}$ **with radius** $r > 0$ is the set

$$S(\mathbf{a}, r) = S_d(\mathbf{a}, r) = D(\mathbf{a}, r) \setminus B(\mathbf{a}, r) = \{\mathbf{x} \in E \mid d(\mathbf{a}, \mathbf{x}) = r\}.$$

Examples:

1. Let $a \in E = \mathbb{R}$ and define $d(x, y) = |x - y|$ for all $x, y \in E$. Then, for $r > 0$, the balls reduce to

$$B(a, r) = \qquad\qquad , \quad D(a, r) = \qquad\qquad ,$$

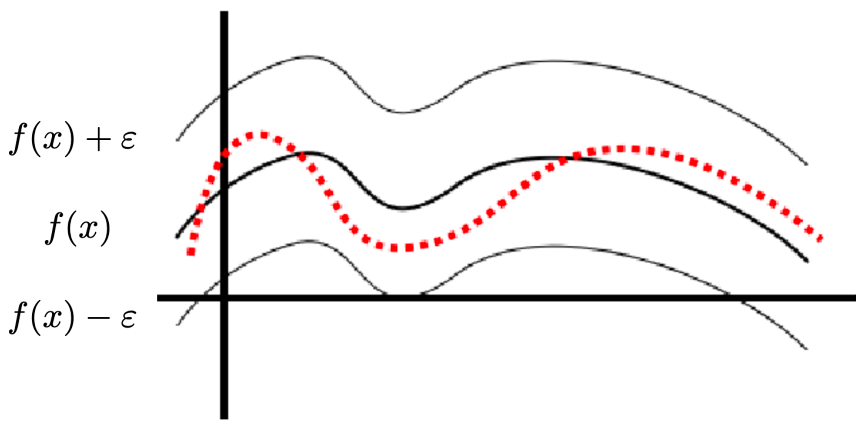
   and the sphere to a discrete set $S(a, r) = \qquad\qquad .$

2. Let (E, d) be a discrete metric space and $\mathbf{a} \in E$. Then

$$B(\mathbf{a}, r) =$$

3. Let $E = C_{\mathbb{R}}([0,1])$, $d_\infty(f,g) = \|f-g\|_\infty$. Then, for $\varepsilon > 0$,

$$B(f,\varepsilon) =$$



4.

Lemma 89. *Let* (E, d) *be a metric space,* $\mathbf{x}, \mathbf{a} \in E$*,* $r > 0$ *and* $\mathbf{x} \notin B(\mathbf{a}, r)$*. Show that* $d(\mathbf{x}, B(\mathbf{a}, r)) \geq d(\mathbf{x}, \mathbf{a}) - r$*.*

Proof.

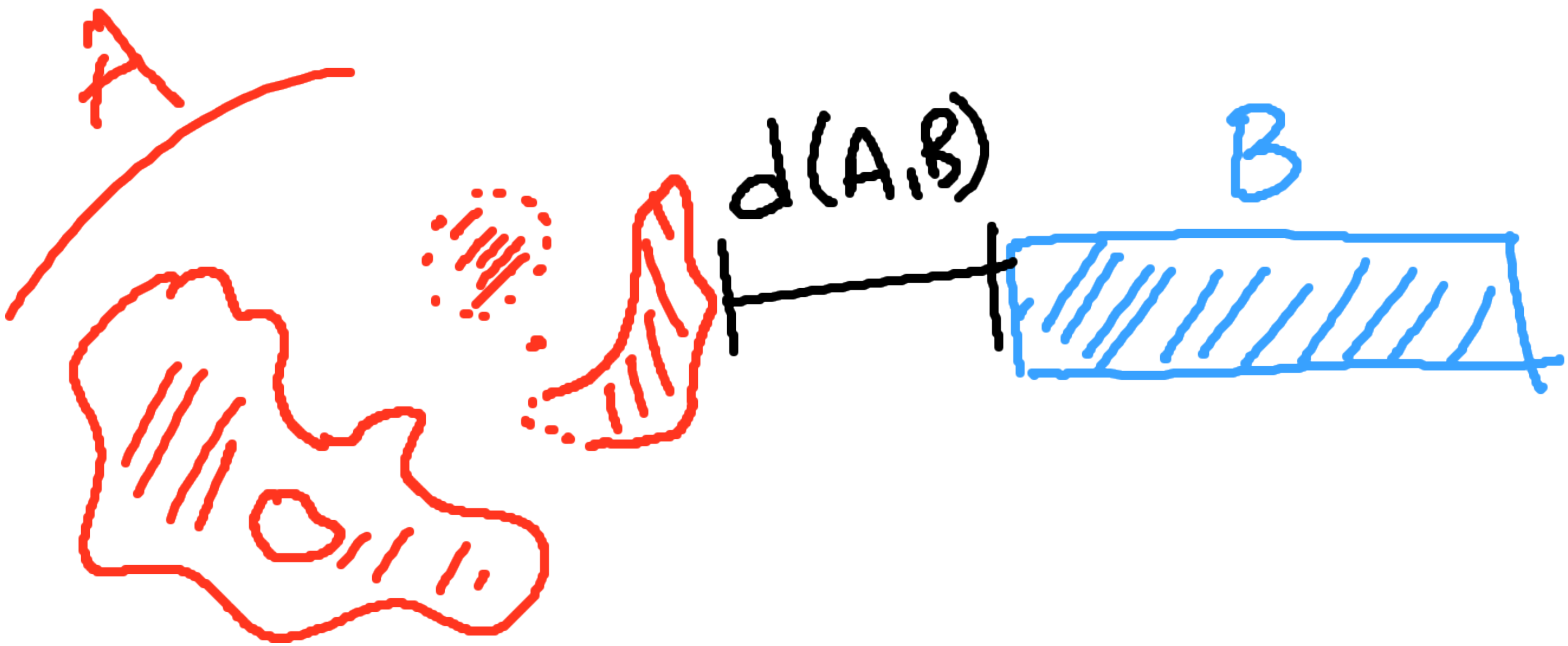
■

Let (E, d) be a metric space and let $\varnothing \neq A \subseteq E$. The **diameter** of A under d is defined by

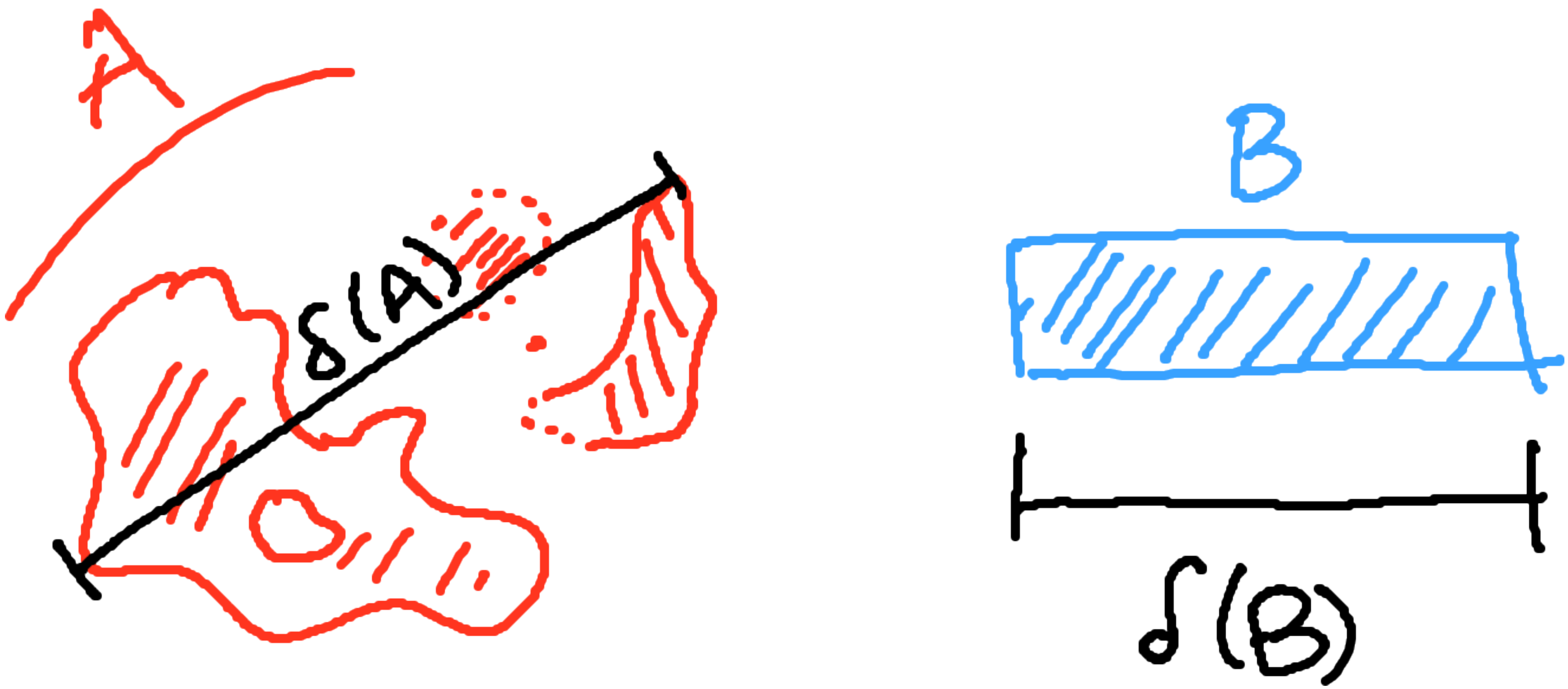
$$\delta_d(A) = \sup_{\mathbf{x},\mathbf{y}\in A} \{d(\mathbf{x}, \mathbf{y})\}.$$

For instance, in $(\mathbb{R}^n, d_2)$, we have $\delta_{d_2}(B(\mathbf{a}, r)) = 2r$.

We say that A is **bounded in** (E, d) if $\delta_d(A) < \infty$.



Distance between two subsets $A, B \subseteq \mathbb{R}^2$.

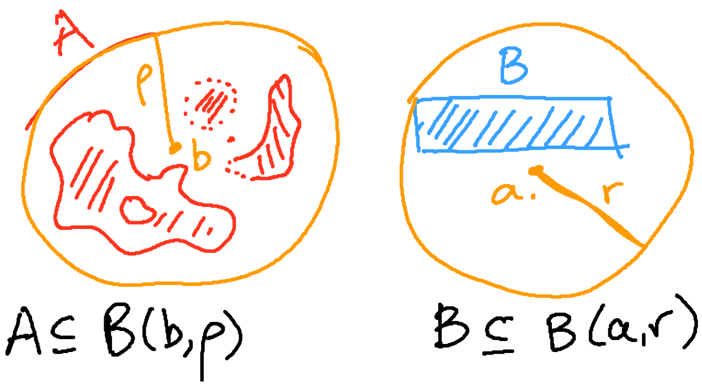


Diameter of two subsets $A, B \subseteq \mathbb{R}^2$.

Proposition 90. *Let (E,d) be a metric space and let $\varnothing \neq A \subseteq E$. Then, A is bounded in (E,d) if and only if $\exists \mathbf{x} \in E$, $\exists r > 0$ such that $A \subseteq B(\mathbf{x}, r)$.*

Proof.

■



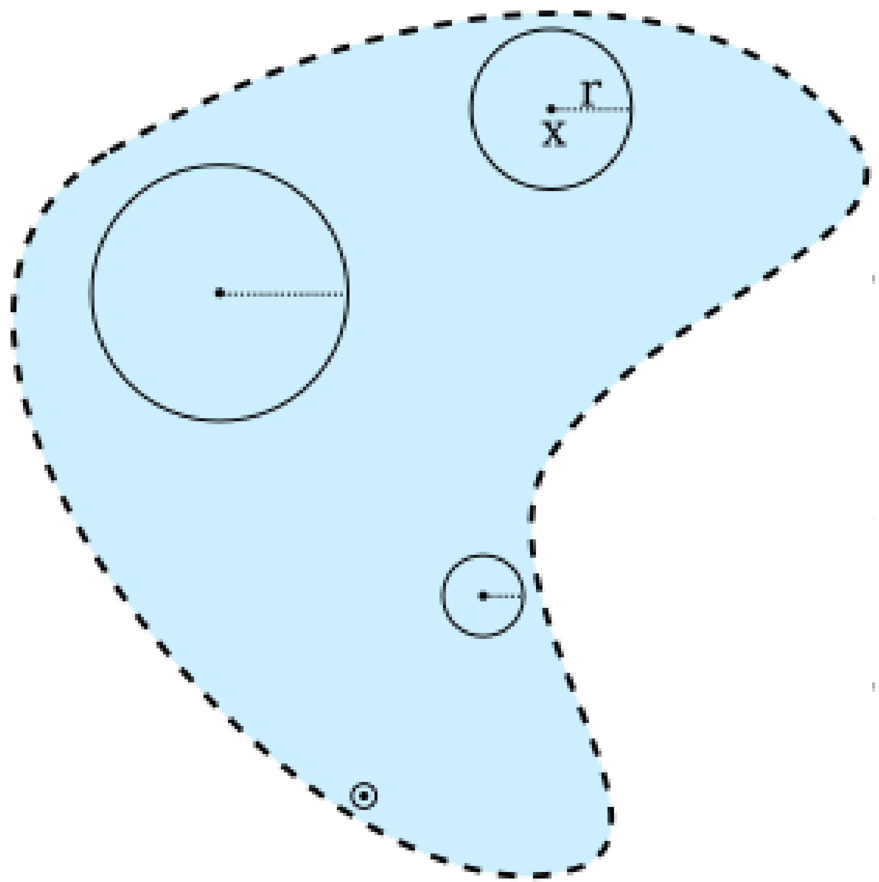
9.1.2 – Metric Space Topology

In this subsection, (E, d) is always a metric space.

A subset $A \subseteq E$ is said to be an **open subset of** E **under** d (or simply to be open if the context is clear) if either

- $A = \varnothing$, or
- $\forall \mathbf{x} \in E$, $\exists r > 0$ such that $B(\mathbf{x}, r) \subseteq A$.

We denote this relationship by $A \subseteq_O E$.



Open subset of $\mathbb{R}^2$ in the Euclidean topology (image from D.J. Eck).

Proposition 91. *Open sets enjoy the following properties:*

1. $E \subseteq_O E$;

2. $\forall \mathbf{a} \in E, r > 0$, *then* $B(\mathbf{a}, r) \subseteq_O E$;

3. *the union of an arbitrary family* $\{A_i\}_{i \in I}$ *of open subsets of* E *is an open subset of* E, *and*

4. *the intersection of a finite family* $\{A_i\}_{i=1}^{\ell}$ *of open subsets of* E *is an open subset of* E.

Proof.

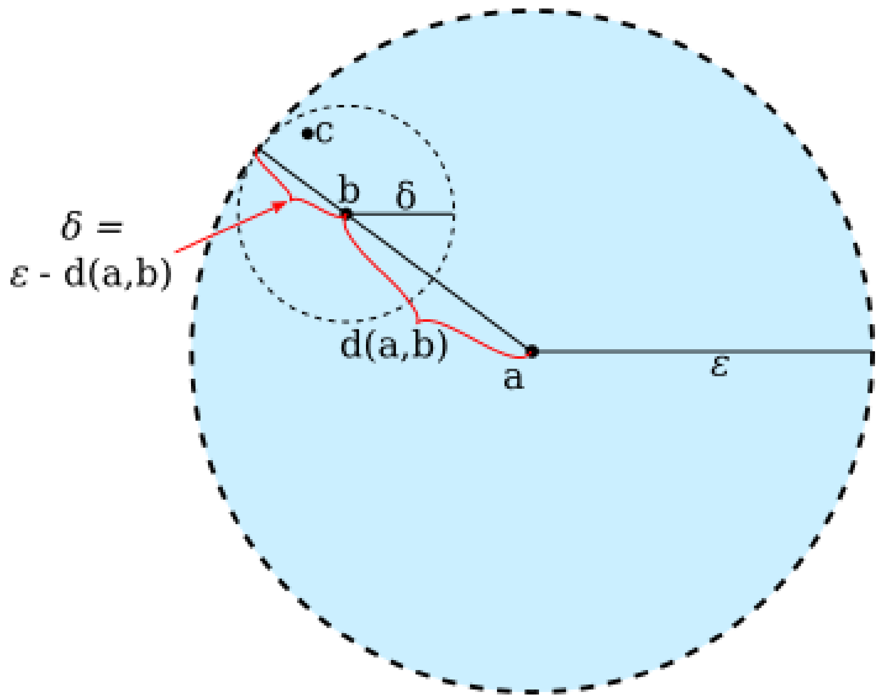
1.

2.

3.

4.

$\blacksquare$



Examples:

1. Let $a \in \mathbb{R}$. Then $(-\infty, a)$ and (a, ∞) are both open in $E = \mathbb{R}$

2. The intersection of an arbitrary family of open subsets of E could be open, but need not be:

   but

A **topology** τ on a set E is a family of subsets of E for which

1. $\varnothing, E \in \tau$
2. if $U_i \in \tau$ for all $i \in I$, then $\bigcup_I U_i \in \tau$
3. if $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$

Examples:

1. Let (E, d) be a metric space. The collection of all open subsets of E under d forms a topology on E, the
2. Let E be any set. The collection $\tau = \{\varnothing, E\}$ forms a topology on E, the

3. Let E be any set. The collection $\tau = \wp(E)$ forms a topology on E, the

A subset $A \subseteq E$ is said to be a **closed subset of E under d** if $E \setminus A \subseteq_O E$. We denote this relationship by $A \subseteq_C E$.

As a consequence of the definition of closed sets in opposition to open sets, we get a whole slew of properties of closed subsets, for free.

1. The empty set is closed in any metric space (E, d).

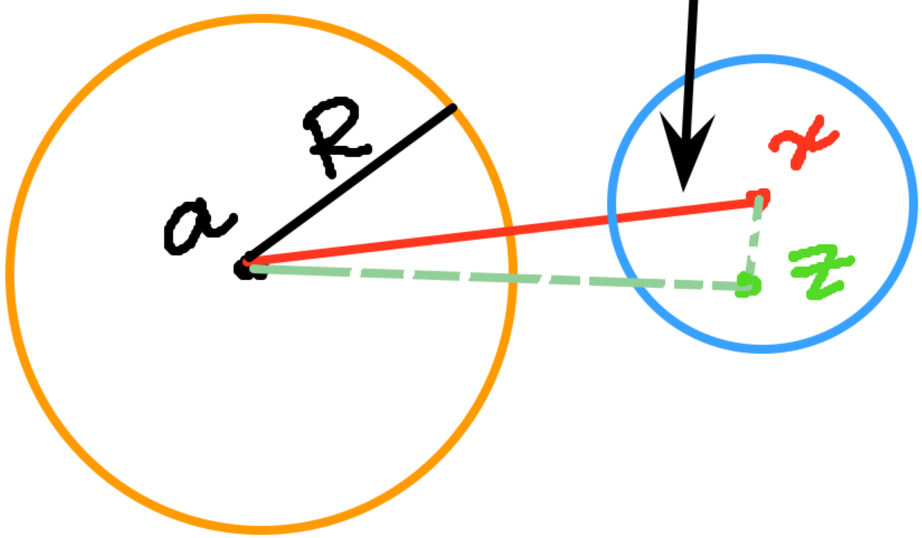
2. Any metric space (E, d) is necessarily a closed subset of itself.

3. Every closed ball in (E, d) is closed.

Proof.

■

$$\rho = \frac{d(a,x) - R}{2} > 0$$



4. Every sphere in (E, d) is closed.

   Proof.

   ■

5. The intersection of an arbitrary family $\{A_i\}_{i\in I}$ of closed subsets of E is a closed subset of E.

6. The union of a finite family $\{A_i\}_{i=1}^{\ell}$ of closed subsets of E is a closed subset of E. Note however that the union of an arbitrary family of closed subsets of E need not be closed (see exercise 17) in E.

The **closure** of a subset $A \subseteq E$ with respect to a metric d is the smallest closed subset $\overline{A}$ of E (again, with respect to d) containing A (with possible equality).

The closure has a number of interesting properties, one of which being that $\overline{A}$ is the intersection of all closed sets containing A, and that $A \subseteq \overline{A}$ (see exercises 18 and 19).

Examples:

1. In the Euclidean topology, $\overline{(0,1)} =$.
2. In the discrete topology, $\overline{(0,1)} =$.
3. In the Euclidean topology, $\overline{S(\mathbf{a},R)} =$.

Lemma 92. *Let A be a subset of E. Then $A \subseteq_C E \iff A = \overline{A}$.*

Proof.

$\blacksquare$

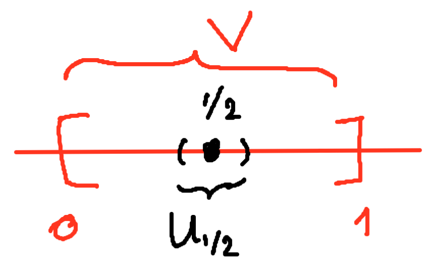
A **neighbourhood** of $\mathbf{x} \in E$ is a subset $V \subseteq E$ containing an open subset $U_{\mathbf{x}} \subseteq_O E$ with $\mathbf{x} \in U_{\mathbf{x}}$. In other words, V is a neighbourhood of $\mathbf{x}$ if $\exists r > 0$ such that $B(\mathbf{x}, r) \subseteq V$ (but V is not necessarily open). The set of all neighbourhoods of $\mathbf{x}$ is denoted by

$$\mathcal{V}(\mathbf{x}) = \{V | V \text{ is a neighbourhood of } \mathbf{x}\}.$$

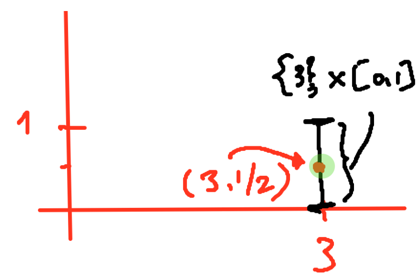
A neighbourhood V of $\mathbf{x}$, with an open set $U_{\mathbf{x}}$.

Examples:

1.



2.



Lemma 93. *Let (E, d) be a metric space with $U \subseteq E$. Then, U is a neighbourhood of each of its points if $U \subseteq_O E$.*

Proof.

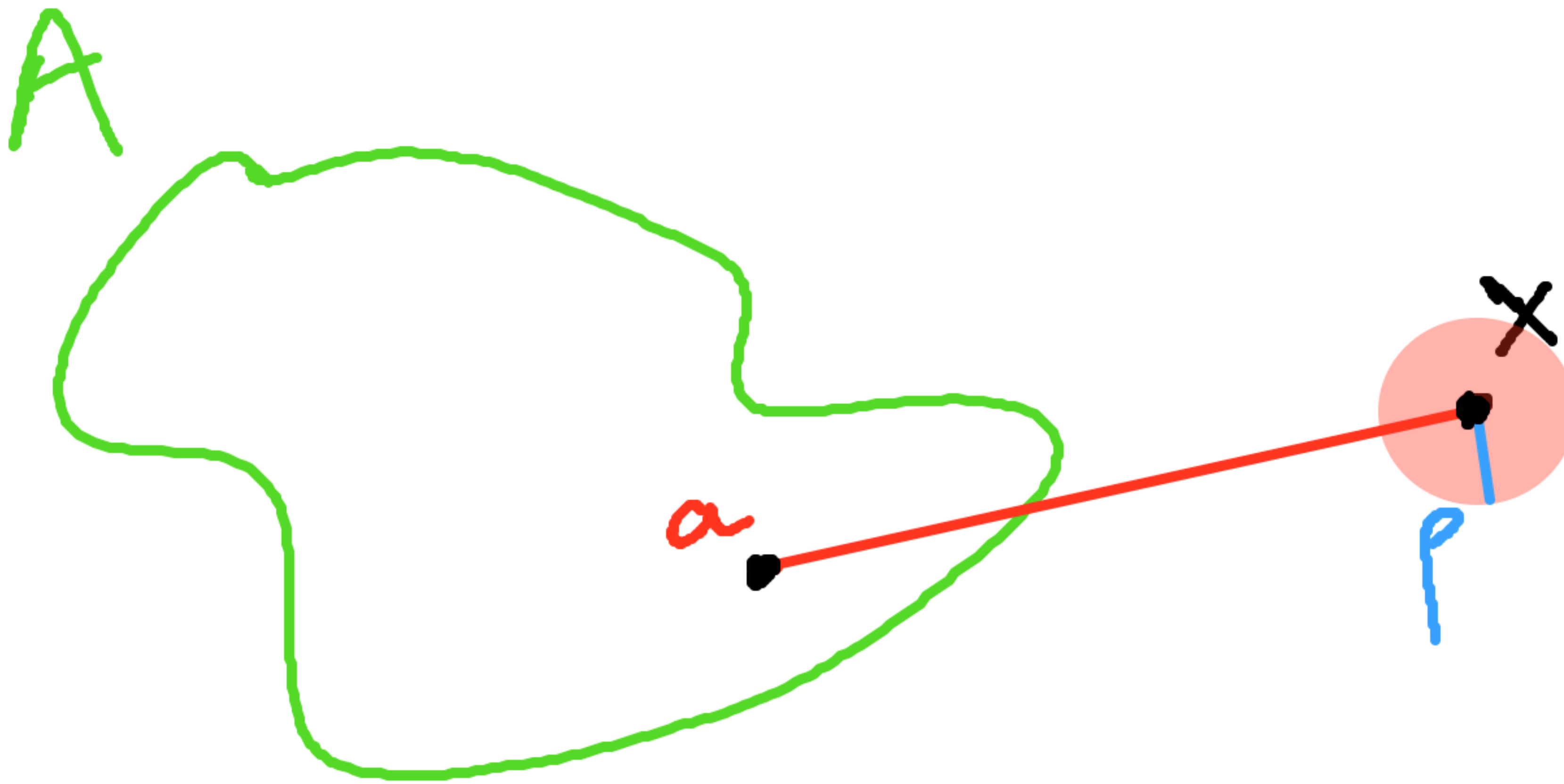
■

Proposition 94. *Let $A \subseteq E$. The following conditions are equivalent:*

1. $\mathbf{x} \in \overline{A}$
2. $\forall \varepsilon > 0$, $\exists \mathbf{a} \in A$ *such that* $d(\mathbf{a}, \mathbf{x}) < \varepsilon$
3. $\forall V \in \mathcal{V}(\mathbf{x})$, $V \cap A \neq \varnothing$
4. $d(\mathbf{x}, A) = 0$

Proof.

∎


A
a
x
ρ

A subset A of E is said to be **dense in** (E, d) if $\overline{A} = E$.

Examples:

1.

2.

3. .

4. **Weierstrass' Theorem:** Let P be the set of polynomial functions $[0, 1] \to \mathbb{R}$. Then P is dense in $(C_{\mathbb{R}}([0, 1]), d_\infty)$. Thus real continuous functions on $[0, 1]$ (which need not even be C^1) can be approximated as closely as desired/needed by smooth (polynomial) functions.

A metric space (E, d) is said to be **separable** if it has at least one dense subset: $\mathbb{R}$ and $\mathbb{R}^n$ are classical examples.

A family $\mathcal{G} = \{G_\lambda\}_{\lambda \in L}$, $\varnothing \neq G_\lambda \subseteq_O E$ forms a **basis for the open subsets of** E if every non-empty open subset of E can be written as a union of members of $\mathcal{G}$.

Examples:

1.

2.

3.

There is a nice way to characterize such bases.

Proposition 95. *A family $\mathcal{G} = \{G_\lambda\}_{\lambda \in L}$ is a basis for the open subsets of E if and only if $\forall \mathbf{x} \in E$, $\forall V \in \mathcal{V}(\mathbf{x})$, $\exists \lambda \in L$ such that $\mathbf{x} \in G_\lambda \subseteq V$.*

Proof.

$\blacksquare$

By analogy with the closure, the **interior** of a subset $A \subseteq E$ is the largest open subset of E contained in A. We denote that subset by $\text{int}(A)$ (or sometimes A°).

It is not hard to show that $\text{int}(A)$ is the union of all the open subsets of E contained in A, and that $A \subseteq_O E$ if and only if $\text{int}(A) = A$ (see exercises).

Examples:

1. In the Euclidean topology, $\text{int}([0, 1]) =$.
2. In the discrete topology, $\text{int}([0, 1]) =$.
3. In the Euclidean topology, $\text{int}(S(\mathbf{a}, R)) =$.

4. In the Euclidean topology, $\text{int}(D(\mathbf{a}, R)) =$.

5. In the Euclidean topology, $\text{int}(\overline{(a, b)}) =$.

6. In the Euclidean topology, $\overline{\text{int}([a, b])} =$.

7. In general, $\text{int}(\overline{W}) \neq W$, as you can see with $W =$.

When $U \subseteq_O E$ and $\text{int}(\overline{U}) = U$, U is a **regular open subset of** E; when $B \subseteq_C E$ and $\overline{\text{int}(B)} = B$, B is a **regular closed subset of** E.

These concepts are not crucial to understand analysis of functions of $\mathbb{R}^n$, but they conform to what our intuition expects of nicely behaved sets.

Not all metrics are derived from a norm (the discrete metric fails in that regard, for instance). **Normed vector spaces** have a very nice property when it comes to closure and balls.

Lemma 96. *If* (E, d) *is a normed vector space, then* $D(\mathbf{0}, 1) = \overline{B(\mathbf{0}, 1)}$*.*

Proof.

■

We can use this lemma to show that the discrete metric is not derived from a norm: were it so, we would have $D(\mathbf{0},1)=\overline{B(\mathbf{0},1)}$. However, in $\mathbb{R}^n$ we have

$$B(\mathbf{0},1)=\{\mathbf{0}\}\subseteq_C \mathbb{R} \text{ and } D(\mathbf{0},1)=\mathbb{R} \implies \overline{B(\mathbf{0},1)}=\{\mathbf{0}\}\neq\mathbb{R}=D(\mathbf{0},1).$$

Proposition 97. *Let $A\subseteq E$. The following conditions are equivalent:*

1. $\mathbf{x}\in\text{int}(A)$

2. $A\in\mathcal{V}(\mathbf{x})$

3. $\exists\varepsilon>0$ *such that* $B(\mathbf{x},\varepsilon)\subseteq A$.

Proof.

■

As an example of the usefulness of this result, note that by the density of $\mathbb{Q}$ and its complement $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$, we automatically get $\text{int}(\mathbb{Q}) = \text{int}(\mathbb{R} \setminus \mathbb{Q}) = \varnothing$ with the usual topology on $\mathbb{R}$.

We end this section with a few other topological concepts. The **boundary** of a subset $A \subseteq E$ is simply defined by $\partial A = \overline{A} \setminus \text{int}(A)$ and the **exterior** of A is given by $\text{int}(E \setminus A)$ (in a nutshell, the exterior is the largest open subset of E which excludes A in its entirety).

We say that $\mathbf{x} \in E$ is a **cluster point** of A if

$$\forall \varepsilon > 0,\ \exists \mathbf{y}_\varepsilon \in B(\mathbf{x}, \varepsilon) \cap A \text{ such that } \mathbf{y}_\varepsilon \neq \mathbf{x}.$$

Finally, we say that $\mathbf{x} \in E$ is an **isolated point** of A if $\exists \varepsilon > 0$ for which $B(\mathbf{x}, \varepsilon) \cap A = \{\mathbf{x}\}$.

Examples: Let $A = \{\frac{1}{n} : n \geq 1\}$.

1. 0 is a cluster point of A since .

2. For all $n \geq 1$, $\frac{1}{n}$ is an isolated point of A, as

Lemma 98. *If $\mathbf{x}$ is a cluster point of A, then $\mathbf{x} \in \overline{A}$ and every neighbourhood of $\mathbf{x}$ contains an infinite set of points in A.*

Proof.

■

Finally, if (E, d) is a metric space and $F \subseteq E$, then (F, d) is also a metric space, called a **metric subspace** of E. The topology on F is completely determined by the topology on E:

Proposition 99. *Let (E, d) be a metric space and $F \subseteq E$. Then*

$$B \subseteq_O F \iff \exists A \subseteq_O E \text{ such that } B = A \cap F$$

$$B \subseteq_C F \iff \exists A \subseteq_C E \text{ such that } B = A \cap F$$

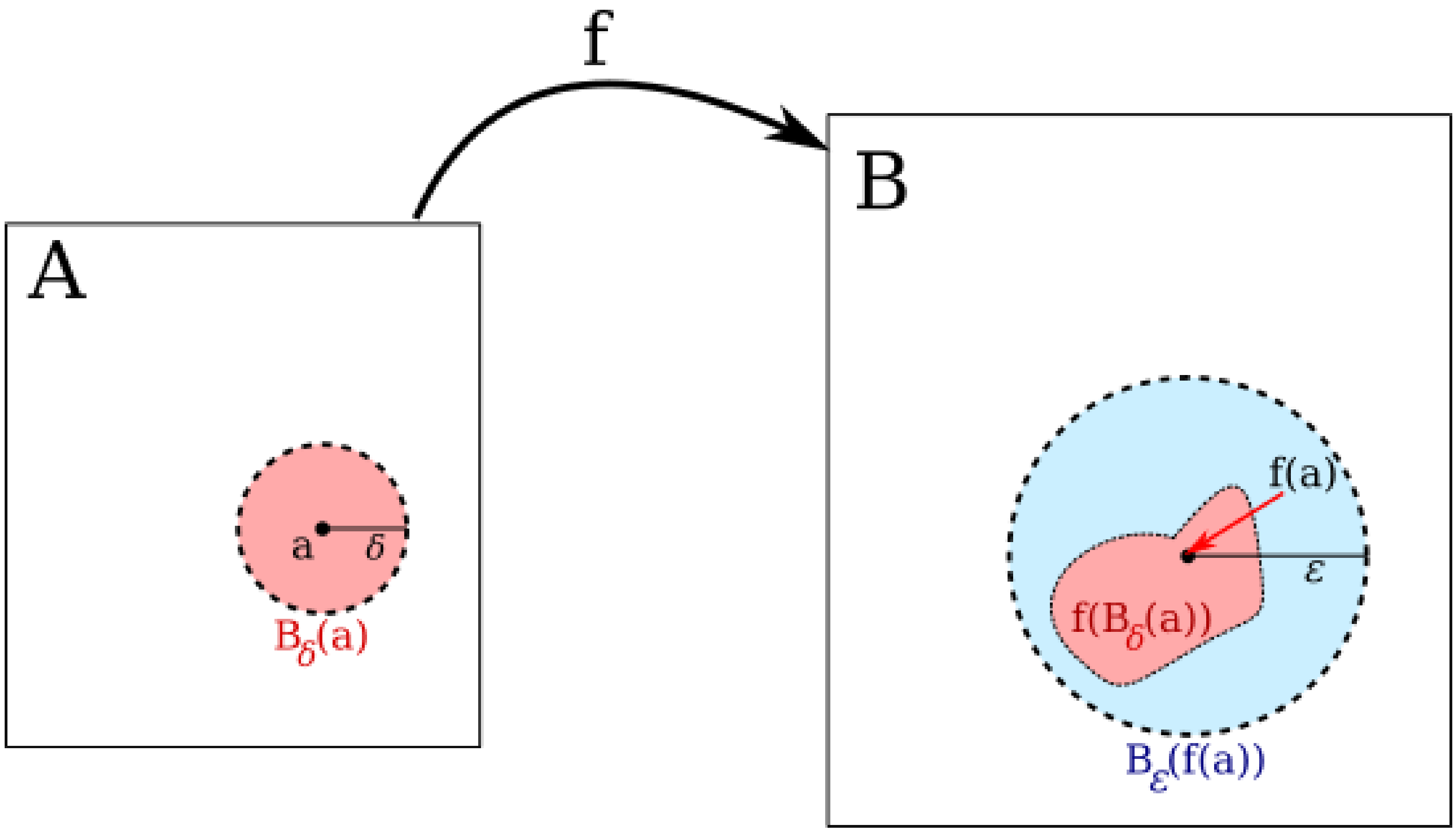
9.1.3 – Continuity

The concept of continuity is fundamental in analysis.

Let $(A, d_A), (B, d_B)$ be metric spaces. As $d_A(\mathbf{a}, \mathbf{x})$ and $d_B(f(\mathbf{a}), f(\mathbf{x}))$ are generalizations of $|a - x|$ and $|f(a) - f(x)|$, respectively, a map $f : A \to B$ is **continuous at** $\mathbf{a} \in A$ if

$$\forall \varepsilon > 0,\ \exists \delta > 0, (\mathbf{x} \in A \text{ and } d_A(\mathbf{a}, \mathbf{x}) < \delta) \implies d_B(f(\mathbf{a}), f(\mathbf{x})) < \varepsilon;$$

or, equivalently, if for any open $\varepsilon-$ball W centered at $f(\mathbf{a})$, there is an open $\delta-$ball V centered at $\mathbf{a}$ such that $f(V) \subseteq W$; or equivalently, if for any neighbourhood $W \subseteq_O B$ of $f(\mathbf{a})$, there is a neighbourhood $V \subseteq_O A$ of $\mathbf{a}$ such that $f(V) \subseteq W$.



Continuity of f at $\mathbf{a} \in \mathbb{R}^2$ in the usual topology (image from D.J. Eck).

That these definitions are equivalent is left as an exercise.

The map f is **continuous on** A if it is continuous at each $\mathbf{a} \in A$.

Proposition 100. *Let* $(E, d), (\tilde{E}, \tilde{d})$ *be metric spaces, and let* $f : E \to \tilde{E}$*. The following conditions are equivalent:*

1. f *is continuous on* E*;*
2. *for any* $W \subseteq_O \tilde{E}$*,* $f^{-1}(W) = \{\mathbf{x} \in E | f(\mathbf{x}) \in W\} \subseteq_O E$*, and*
3. *for any* $Y \subseteq_C \tilde{E}$*,* $f^{-1}(Y) \subseteq_C E$*.*

Proof.

■

Consider a map $f : E \to \tilde{E}$ as above. If $f(W) \subseteq_O \tilde{E}$ for all $W \subseteq_O E$, then we say that f is an **open mapping**; by analogy, if $f(Y) \subseteq_C \tilde{E}$ for all $Y \subseteq_C E$, then we say that f is a **closed mapping**.

Generally speaking, continuous maps are neither open nor closed; the constant function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a$ provides an example of a continuous function which is not open in the standard topology, as $(0,1) \subseteq_O \mathbb{R}$, but $f((0,1)) = \{a\} \subseteq_C \mathbb{R}$, for instance.

Proposition 101. *Let* $f : (E,d) \to (\tilde{E},\tilde{d})$ *and* $g : (\tilde{E},\tilde{d}) \to (\hat{E},\hat{d})$ *be continuous. Then the composition* $g \circ f : (E,d) \to (\hat{E},\hat{d})$ *is continuous.*

Proof.

■

In many instances, the broad strokes of proofs in the multi-dimensional cases follow those of the one-dimensional proofs.

Corollary 102. *Let* $f : (E, d) \to (\tilde{E}, \tilde{d})$ *be a continuous function. If* $F \subseteq E$*, then the restriction* $f|_F : (F, d|_F) \to (\tilde{E}, \tilde{d})$ *is continuous.*

Proof.

■

Examples:

1.

2.

3.

4.

A metric d on E gives rise to a topology by defining the open sets of E. A natural question to ask is: can two different metrics could give rise to the same topology? In order to answer that question, we need to introduce a new concept.

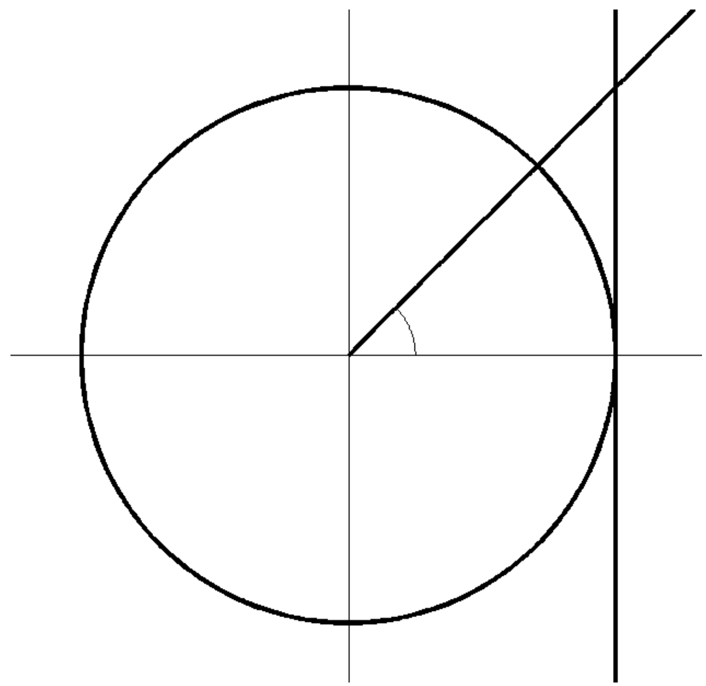
Let $(E, d), (\tilde{E}, \tilde{d})$ be metric spaces. A function $f : E \to \tilde{E}$ is a **homeomorphism** if f is bijective and both f and f^{inv} are continuous (alternatively, f is a homeomorphism if it is bijective, continuous and open).

Examples:

1.

2.

3.



These examples illustrate that the notion of boundedness is not necessarily preserved by homeomorphisms: for instance, $\mathbb{R}$ is unbounded while $(-\frac{\pi}{2}, \frac{\pi}{2})$ is bounded, but both spaces are homemorphic to one another *via* the $\arctan$.

Furthermore, neither is the notion of distance necessarily preserved by homeomorphisms: in general,

$$d(x_1, x_2) \neq \tilde{d}(f(x_1), f(x_2)).$$

For instance, in the first example,

$$d(0,2) = |0-2| = 2 \neq \tilde{d}(0^3, 2^3) = |0^3 - 2^3| = 9.$$

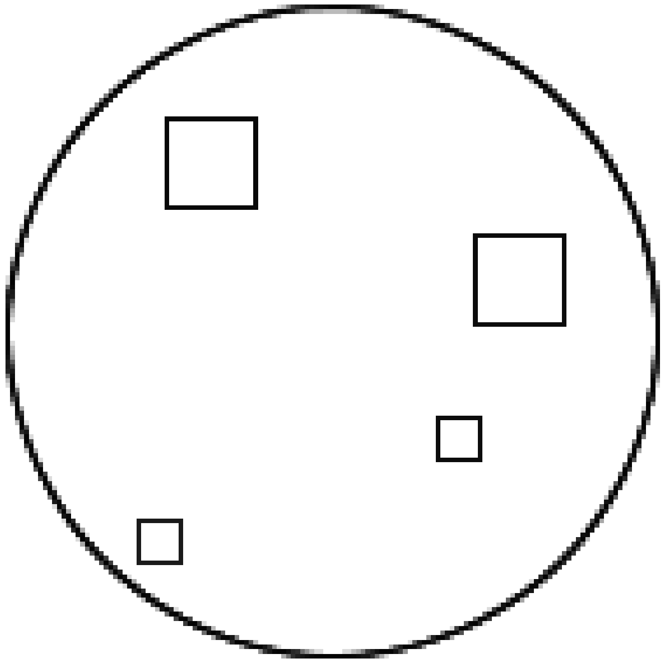
However, homeomorphisms $f : E \to \tilde{E}$ **preserve the topologies of E and $\tilde{E}$:**

$$W \subseteq_O E \iff f(W) \subseteq_O \tilde{E} = f(E)$$

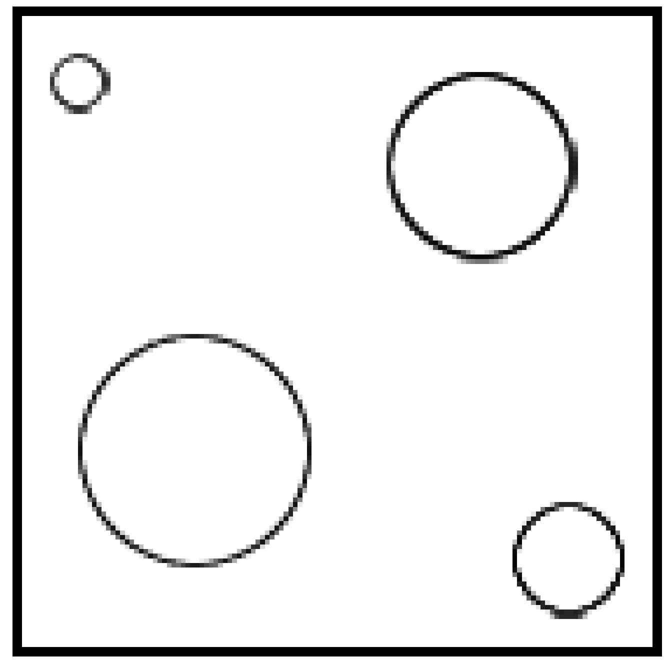
$$Y \subseteq_C E \iff f(Y) \subseteq_C \tilde{E} = f(E).$$

Two metrics $d, \tilde{d}$ on E are said to be **topologically equivalent** if $\mathrm{id} : (E, d) \to (E, \tilde{d})$ is a homeomorphism. In that case, d and $\tilde{d}$ give rise to the same topologies on E.

Example: if $p, q \geq 1$, d_p and d_q induce the same topologies on $\mathbb{R}^n$.



$2-$ball filled with with $\infty-$balls



$\infty-$ball filled with with $2-$balls

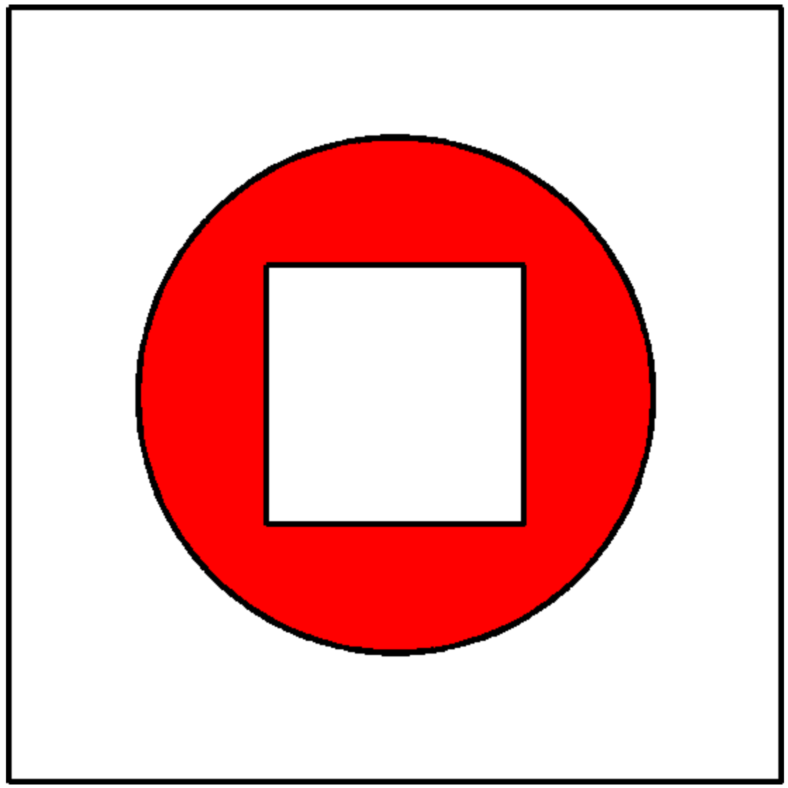
Two metrics $d, \tilde{d}$ on E are **(strongly) equivalent** if $\exists A, B > 0$ such that

$$Ad(\mathbf{x}, \mathbf{y}) \leq \tilde{d}(\mathbf{x}, \mathbf{y}) \leq Bd(\mathbf{x}, \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in E.$$

Intuitively, two metrics are equivalent if it is always possible to fit a $\tilde{d}-$ball between two $d-$balls, while maintaining the ratios of the balls' radii.

Clearly, if two metrics are equivalent on E, they must also be topologically equivalent, but the converse may not always hold (see exercise 43).

Example: if $p, q \geq 1$, d_p and d_q are equivalent on $\mathbb{R}^n$.



Given the geometry of squares and circles, what values can A and B take?

There is also a similar notion for norms. Two norms $\|\cdot\|^*$, $\|\cdot\|^\circ$ on E are **equivalent** if $\exists a, b > 0$ such that

$$a\|\mathbf{x}\|^* \leq \|\mathbf{x}\|^\circ \leq b\|\mathbf{x}\|^*, \quad \forall \mathbf{x} \in E.$$

Clearly, two equivalent norms on E give rise to two equivalent metrics on E. Over a **finite**−dimensional vector space, any two norms are equivalent:

1. WLOG, assume $\|\cdot\|^* = \|\cdot\|_1$;
2. only vectors $\mathbf{x} \in S_1(\mathbf{0}, 1)$ need to be considered;
3. show that $\|\cdot\|^\circ$ is continuous with respect to $\|\cdot\|_1$, and
4. use the Max/Min Theorem over $S_1(\mathbf{0}, 1)$ to bound $a \leq \|\mathbf{x}\|^\circ \leq b$.

We end this section on preliminaries with two definitions that generalize the notion of a continuous function.

Let $f : (E, d) \to (\tilde{E}, \tilde{d})$. We say that f is

1. **uniformly continuous** if $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that $\forall \mathbf{x}, \mathbf{y} \in E$, $d(\mathbf{x}, \mathbf{y}) < \delta \implies \tilde{d}(f(\mathbf{x}), f(\mathbf{y})) < \varepsilon$;

2. **Lipschitz continuous** if $\exists K > 0$ such that $\tilde{d}(f(\mathbf{x}), f(\mathbf{y})) \leq K d(\mathbf{x}, \mathbf{y})$.

The conceptual difference between continuity and uniform continuity is that δ may depend on $\mathbf{x}$ and $\mathbf{y}$ as well as ε if the function is continuous, but it can only depend on ε for uniformly continuous functions.

Examples:

1.

2.

3.

4.

Two metrics $d, \tilde{d}$ on E are **uniformly equivalent** if $\mathrm{id} : (E, d) \to (E, \tilde{d})$ is uniformly continuous, and so is its inverse.

Uniformly equivalent metrics are topologically equivalent, as uniform continuity also implies continuity, but there are topologically equivalent metrics that are not uniformly equivalent. However, uniform equivalence and strong equivalence of metrics are ... well, equivalent.

Lastly, note that uniform continuity, unlike continuity, is not a **topological notion**: given a function $f : E \to \tilde{E}$, the knowledge of the topologies on E and $\tilde{E}$, respectively, is sufficient to determine if f is continuous.

But more must be known in order to determine if f is uniformly continuous. There is something fundamental at play here; we will return to it at a later stage.

9.2 – Sequences in a Metric Space

Consider the sequence $(\mathbf{x}_n) \subseteq (E,d)$. The sequence **converges** to $\mathbf{x} \in (E,d)$, denoted by $\mathbf{x}_n \to \mathbf{x}$, if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n > N \implies d(\mathbf{x}_n, \mathbf{x}) < \varepsilon.$$

In light of the notions presented in the previous section, this is equivalent to the following definition: $\mathbf{x}_n \to \mathbf{x} \in E$ if

$$\forall V \in \mathcal{V}(\mathbf{x}),\ \exists N \in \mathbb{N} \text{ such that } n > N \implies \mathbf{x}_n \in V.$$

Thus a sequence converges to $\mathbf{x}$ if any neighbourhood of $\mathbf{x}$ contains infinitely many terms in the sequence.

A **subsequence of** $(\mathbf{x}_n)$ is a sequence $(\mathbf{y}_n)$ such that $\mathbf{y}_n = \mathbf{x}_{\varphi(n)}$ for some strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$. It is easy to show that if $\mathbf{x}_n \to \mathbf{x}$, then any subsequence of $(\mathbf{x}_n)$ also converges to $\mathbf{x}$ (see the exercises).

Let $(\mathbf{x}_n)$ be a sequence in a metric space (E, d). We say that $\mathbf{a} \in E$ is a **limit point of** $(\mathbf{x}_n)$ if $\forall \varepsilon > 0$, $\forall \rho \in \mathbb{N}$, $\exists n \geq \rho$ such that $d(\mathbf{x}_n, \mathbf{a}) < \varepsilon$.

Proposition 103. *Let $(\mathbf{x}_n) \subseteq (E, d)$, $\mathbf{a} \in E$. The following are equivalent:*

1. *$\mathbf{a}$ is a limit point of $(\mathbf{x}_n)$;*
2. *there is a subsequence of $(\mathbf{x}_n)$ which converges to $\mathbf{a}$;*
3. *$\forall \rho \in \mathbb{N}$, we have $\mathbf{a} \in \overline{A_\rho}$, where $A_\rho = \{\mathbf{x}_n | n \geq \rho\}$, and*
4. *either $\mathbf{a}$ is a cluster point of A_1 or $\{\mathbf{x}_n \mid \mathbf{x}_n = \mathbf{a}\}$ is infinite.*

If $\{\mathbf{x}_n \mid \mathbf{x}_n = \mathbf{a}\}$ is infinite, $\mathbf{a}$ is a **replicating point of** $(\mathbf{x}_n)$.

Proof. We prove $1. \implies 2. \implies 3. \implies 4. \implies 1.$

■

9.2.1 – Closure, Closed Subsets and Continuity

We can conclude from Proposition 103 that the set $\bigcap_{\rho\in\mathbb{N}} \overline{A_\rho}$ of limit points of $(\mathbf{x}_n)$ is closed and that if $\mathbf{x}_n \to \mathbf{x}$, then $\mathbf{x}$ is the unique limit point of $(\mathbf{x}_n)$.

There is a nice way to characterize closure, closed subsets and continuity using sequences and convergence.

Proposition 104. *Let (E,d) be a metric space, $A \subseteq E$ and $\mathbf{x} \in E$. Then,*

$$\mathbf{x} \in \overline{A} \Longleftrightarrow \exists (\mathbf{x}_n) \subseteq A \text{ such that } \mathbf{x}_n \to \mathbf{x}.$$

Proof.

$\blacksquare$

Proposition 105. *Let (E, d) be a metric space, with $F \subseteq E$. Then, $F \subseteq_C E$ if and only if any sequence $(\mathbf{x}_n) \subseteq F$ which converges in E converges to a point in F.*

Proof.

$\blacksquare$

Proposition 106. *Let* $(E, d), (\tilde{E}, \tilde{d})$ *be a metric spaces. Then* $f : E \to \tilde{E}$ *is continuous if and only* $f(\mathbf{x}_n) \to f(\mathbf{x})$ *whenever* $\mathbf{x}_n \to \mathbf{x}$*.*

Proof.

$\blacksquare$

9.2.2 – Complete Spaces and Cauchy Sequences

The sequence $(\mathbf{x}_n) \subseteq (E, d)$ is a **Cauchy sequence** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } n, m > N \implies d(\mathbf{x}_n, \mathbf{x}_m) < \varepsilon.$$

Some properties of Cauchy sequences in $\mathbb{R}$ carry over to metric spaces.

Proposition 107. *Convergent sequences in (E, d) are Cauchy.*

Proof.

■

In a normed space $(E, \|\cdot\|)$, a sequence $(\mathbf{x}_n)$ is **bounded** if $\exists M \in \mathbb{N}$ such that $\|\mathbf{x}_n\| < M$ for all $n \in \mathbb{N}$.

But a metric space (E, d) is not necessarily a normed vector space, so there might not be a norm to use to determine boundedness.

In a general metric space (E, d), a sequence $(\mathbf{x}_n)$ is **bounded** if $\exists M > 0$ s.t. $\mathbf{x}_n \in B(\mathbf{0}, M)$ for all $n \in \mathbb{N}$. Similarly, $A \subseteq E$ is **bounded** if $\delta(A) < \infty$.

Proposition 108. *Every Cauchy sequence in (E, d) is bounded.*

Proof.

■

But the notion of a Cauchy sequence is not topological.

Example: Let $A = (0, \infty)$. Consider the following metrics on A:

$$d_1(x, y) = |x - y| \quad \text{and} \quad d_2(x, y) = |\ln x - \ln y|.$$

Show that both metrics induce the same topology on A, but that Cauchy sequences under one are not necessarily Cauchy sequences under the other.

Solution.

■

This could not happen, however, if the metrics are strongly equivalent, which further illustrates the distinctness of the notions of strong equivalence and topological equivalence (and also justifies the use of the "strong" qualifier).

Proposition 109. *Let* d *and* $\tilde{d}$ *be two equivalent metrics on* E. *Then,* $(\mathbf{x}_n)$ *is a Cauchy sequence in* (E, d) *if and only if* $(\mathbf{x}_n)$ *is a Cauchy sequence in* $(E, \tilde{d})$.

Proof.

∎

A metric space (E, d) is said to be **complete** if every single one of its Cauchy sequences is convergent. If a complete metric space is also a normed vector space, then it is said to be a **Banach space**. If a Banach space is also an inner product space, then it is said to be a **Hilbert space**.

Examples: (COMPLETE, BANACH AND HILBERT SPACES)

1.

2.

3.

4.

Closed subsets of complete spaces are especially well-behaved.

Proposition 110. *Every closed subset of a complete metric space is complete.*

Proof.

■

Proposition 111. *Every complete subspace of a metric space is closed.*

Proof.

■

Proposition 112. *Let (E_i, d_i) be metric spaces for $i = 1, \ldots, n$. The metric space $(E, d) = (E_1 \times \cdots \times E_n, \sup_{i=1,\ldots,n}\{d_i\})$ is complete if and only if (E_i, d_i) for all $i = 1, \ldots, n$.*

Proof. ■

The following result is a generalization of the Nested Intervals Theorem.

Proposition 113. *Let (E, d) be a complete metric space. If (F_n) is a decreasing sequence of non-empty closed subsets of E*

$$E \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$$

such that $\lim_{n\to\infty} \delta(F_n) = 0$, then $\bigcap_{n\geq 1} F_n = \{\mathbf{x}\}$ for some $\mathbf{x} \in E$.

Proof.

∎

The following result about contractions is representative of a family of extremely useful theorems.

Theorem 114. (Fixed Point Theorem)
Let (E, d) be a a complete metric space and let $f : E \to E$ be a ***contraction*** *on E, that is,*

$$\exists k \in (0,1) \text{ such that } d(f(\mathbf{x}), f(\mathbf{y})) \leq kd(\mathbf{x}, \mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in E.$$

Then $\exists! \mathbf{x}^* \in E$ such that $f(\mathbf{x}^*) = \mathbf{x}^*$; $\mathbf{x}^*$ is a ***fixed point of*** *f.*

Proof.

∎

But the choice of $\mathbf{x}_0 \in E$ was arbitrary. If f is a contraction, the sequence $(f^n(\mathbf{x}))$ converges to the unique fixed point for all $\mathbf{x} \in E$.

The restriction $k \in (0,1)$ is necessary, as the following example demonstrates.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & x < 0 \\ x + \frac{1}{x+1}, & x \geq 0 \end{cases}.$$

It is not hard to see that f has no fixed point (see exercise 62), yet

$$d(f(x), f(y)) \leq d(x,y) \quad \text{for all } x, y \in \mathbb{R}.$$

9.3 – Exercises

1. Show that the absolute value defines a norm on $\mathbb{R}$.
2. Show that the modulus defines a norm on $\mathbb{C}$.
3. Show that the sup norm $\|\cdot\|_\infty$ is indeed a norm on $C_\mathbb{R}([0,1])$.
4. Let $\infty \geq p \geq 1$. Show that the $p-$norm $\|\cdot\|_\infty$ is indeed a norm on $\mathbb{R}^n$.
5. Let $p \geq 1$. Show that (1, p. 9), defines a norm on $\mathcal{L}^p([0,1])$.
6. Prove Lemma 88, p. 11.
7. Let E be any set. Show that (2, p. 13) defines a metric on E.
8. Let $E = \mathbb{R}^n$. Show that d_2 is a metric on E.
9. Let $E = \mathbb{R}$, $d(x,y) = |x-y|$, $A = \mathbb{N}$ and $B = \{\frac{n-1}{n} \mid n \in \mathbb{N}\}$. Compute $d(A,B)$, where d is as in (3, p. 17)). Can you use this result to show that (3, p. 17) does not define a metric on $\wp(E) \setminus \varnothing$?
10. In a metric space, show that $\delta(A) \in [0,\infty]$. Also, show that $\delta(A) = 0 \iff A$ is a singleton.

11. Prove or disprove: In any metric space (E,d), $\delta_d(B(\mathbf{a},r)) = 2r$.
12. Prove or disprove: Let d, d' be metrics on E. Then, A is bounded in (E,d) if and only if A is bounded in (E,d').
13. Where does the proof that a finite intersection of open subsets is open fail for arbitrary intersections?
14. Show that the metric space topology on a discrete metric space is the discrete topology.
15. Show that the intersection of an arbitrary family $\{A_i\}_{i\in I}$ of closed subsets of E is a closed subset of E.
16. Show that the union of a finite family $\{A_i\}_{i=1}^{\ell}$ of closed subsets of E is a closed subset of E.
17. Show that the union of an arbitrary family of closed subsets of E need not be closed in E.
18. Let A be a subset of a metric space (E,d). Show that $\overline{A}$ is the intersection of all closed subsets of E containing A.
19. Let A be a subset of a metric space (E,d). Show that $A \subseteq \overline{A}$.
20. Prove Lemma 92, p. 34.

21. In Proposition 94, p. 37, show that $2. \iff 3 \iff 4.$
22. Let A be a subset of a metric space (E, d). Show that $\mathrm{int}(A)$ is the union of all open subsets of E contained in A.
23. Let A be a subset of a metric space (E, d). Show that $\mathrm{int}(A) \subseteq A$.
24. Let A be a subset of a metric space (E, d). Show that $A \subseteq_O E \iff A = \mathrm{int}(A)$.
25. Let A, B be subsets of a metric space (E, d). Show that
    (a) $B \subseteq A \implies \mathrm{int}(B) \subseteq \mathrm{int}(A)$
    (b) $B \subseteq A \implies \overline{B} \subseteq \overline{A}$
    (c) $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$
    (d) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
    (e) $\mathrm{int}(A) \cup \mathrm{int}(B) \subseteq \mathrm{int}(A \cup B)$
    (f) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$
26. In each instance, give an example showing that, in general,
    (a) $\mathrm{int}(A) \cup \mathrm{int}(B) \neq \mathrm{int}(A \cup B)$
    (b) $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$

27. Let A be subset of a metric space (E,d). Show that
    (a) $E\setminus \text{int}(A)=\overline{E\setminus A}$
    (b) $E\setminus \overline{A}=\text{int}(E\setminus A)$
    (c) $\partial(\text{int}(A))\subseteq \partial A$
    (d) $\partial\overline{A}\subseteq \partial A$
28. Find an example of a subset A of a metric space (E,d) for which $\partial(\text{int}(A))$, ∂A and $\partial\overline{A}$ are all different.
29. Find two subsets $A,B\subseteq (R,d_2)$ for which $A\cup B$, $\text{int}(A)\cup B$, $A\cup \text{int}(B)$, $\text{int}(A)\cup\text{int}(B)$, and $\text{int}(A\cup B)$ are all distinct.
30. Find a subset $A\subseteq (R,d_2)$ for which A, $\text{int}(A)$, $\overline{A}$, $\text{int}(\overline{A})$, $\overline{\text{int}(A)}$, $\overline{\text{int}(\overline{A})}$ and $\text{int}\left(\overline{\text{int}(A)}\right)$ are all distinct.
31. For any subset $A\subseteq (R,d_2)$, show that $\text{int}\left(\overline{\text{int}(\overline{A})}\right)=\text{int}(\overline{A})$.
32. Complete the proof of Lemma 98, p. 48.
33. Prove Proposition 99, p. 49.

34. We say that $A \subseteq E$ is **meagre** (or nowhere dense) if and only if $\mathrm{int}(\overline{A}) = \varnothing$. Show that
    (a) A is meagre if and only if $\mathrm{int}(E \setminus A)$ is dense in E (a set A is **dense** in B if $A \subseteq B \subseteq \overline{A}$);
    (b) A is meagre if and only if A is contained in a closed subset of E whose interior is empty;
    (c) A is closed and meagre if and only if $A = \partial A$, and
    (d) A is meagre $\implies$ $\overline{A} = \partial A$.
35. Show that the three definitions of continuity are equivalent.
36. Let $f : C \to D$, $A \subseteq C$ and $B \subseteq D$. Show that $f^{-1}(f(A)) = A$ and that in general, the best we can say is that $f(f^{-1}(B)) \subseteq B$.
37. Can you find a function $f : E \to \tilde{E}$ which is continuous but not closed?
38. Can you find a function $f : E \to \tilde{E}$ which is open and closed but not continuous?
39. Can you find a function $f : E \to \tilde{E}$ which is open and continuous but not closed?
40. Complete the proof of Proposition 101, p. 54.
41. Complete the proof of Corollary 102, p. 55.

42. Provide the details showing that d_2 and d_∞ are topologically equivalent on $\mathbb{R}^2$.
43. Consider the metric space $(\mathbb{R}, d_2)$. Define a new function $\tilde{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \to \mathbb{R}$ by

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

    Show that $\tilde{d}$ defines a metric on $\mathbb{R}$, that d and $\tilde{d}$ are topologically equivalent but that they are not equivalent.
44. Let (E, d) be a metric space. Show that $d : E \times E \to \mathbb{R}$ is Lipschitz continuous (with $k = 2$) and so that it is a continuous map.
45. Find a function which is uniformly continuous but not Lipschitz continuous.
46. Show that the two definitions of convergence of a sequence are equivalent.
47. Show that if $\mathbf{x}_n \to \mathbf{x}$, then any subsequence of $(\mathbf{x}_n)$ also converges to $\mathbf{x}$.
48. Show that the set of limit points of a sequence is closed.
49. Complete the proof of Proposition 103, p. 69.
50. Prove Proposition 112, p. 84.

51. Show that d_∞, d_1 and d_2 are equivalent on $\mathbb{R}^2$.
52. For $i = 1, \ldots, n$, let (E_i, d_i) be metric spaces and $U_i \subseteq_O E_i$. Show that $U_1 \times \cdots \times U_n$ is an open subset of

$$(E, d) = (E_1 \times \cdots \times E_n, \sup\{d_i \mid i = 1, \ldots, n\})$$

53. For $i = 1, \ldots, n$, let (E_i, d_i) be metric spaces and let $\pi_i : E_1 \times \cdots \times E_n \to E_i$ be defined by $\pi_i(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{x}_i$. Show that π_i is open and continuous.
54. Show that a map $f : (F, \delta) \to (E_1, d_1) \times \cdots \times (E_n, d_n)$ is continuous at $\mathbf{a} \in F$ if and only if $\pi_i \circ f$ is continuous at $\mathbf{a} \in F$ for all i.
55. Let $f : (E_1, d_1) \times \cdots \times (E_n, d_n) \to (F, \delta)$ and $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in E$. For all i, define $f_i : (E_i, d_i) \to (F, \delta)$ by $f_i(\mathbf{x}) = f(\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{x}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n)$. Show that if f is continuous at $\mathbf{a}$, then f_i is continuous at $\mathbf{a}$ for all i.
56. Show that $d = \sup\{d_i \mid i = 1, \ldots, n\}$ defines a metric on $E = \prod_{i=1}^n (E_i, d_i)$.
57. Let (E_i, d_i) be metric spaces for $i = 1, \ldots, n$. Show that the metric prouct space $(E, d) = (\prod E_i, \sup\{d_i\})$ is complete if and only if (E_i, d_i) is complete for each i.

58. Show that the converse of the previous result does not hold in general, for instance for $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & \text{else} \end{cases}$$

59. Let $d_1, d_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be defined according to

$$d_1(m, n) = \begin{cases} 0, & \text{if } m = n \\ 1 + \frac{1}{m+n}, & \text{otherwise} \end{cases} \quad d_2(m, n) = \frac{|m - n|}{mn}.$$

   (a) Show that d_1 and d_2 are metrics on $\mathbb{N}$.
   (b) Show that the topologies of $(\mathbb{N}, d_1)$ and $(\mathbb{N}, d_2)$ are both discrete.
   (c) Show that $(\mathbb{N}, d_1)$ is complete but that $(\mathbb{N}, d_2)$ is not.
   (d) What does this say about completeness as a topological property of a space?

60. Show that the space $\ell^2(\mathbb{N})$ is a Hilbert space as follows.
    (a) Show that $\ell^2(\mathbb{N})$ is a vector space over $\mathbb{C}$.
    (b) Show that $(\cdot|\cdot)$ defined in the text is indeed an inner product over $\ell^2(\mathbb{N})$.
    (c) Show that $(\cdot|\cdot)$ defines a norm $\|\cdot\|$ over $\ell^2(\mathbb{N})$.
    (d) Show that $\ell^2(\mathbb{N})$ is complete under $\|\cdot\|$.
61. Let (E, d) be a metric space. Define $d_1, d_2 : E \times E \to \mathbb{R}$ by $d_1(\mathbf{x}, \mathbf{y}) = \frac{d(\mathbf{x},\mathbf{y})}{1+d(\mathbf{x},\mathbf{y})}$ and $d_2(\mathbf{x}, \mathbf{y}) = \min\{d(\mathbf{x}, \mathbf{y}), 1\}$.
    (a) Show that d_1 and d_2 are metrics on E.
    (b) Show that d is topologically equivalent to d_2.
    (c) Show that d_1 is topologically equivalent to d_2.
62. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & x < 0 \\ x + \frac{1}{x+1}, & x \geq 0 \end{cases}.$$

Show that f has no fixed point but that $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in \mathbb{R}$.

63. Let X be a compact metric space. Define

$$C_{\mathbb{R}}(X) = \{f | f : X \to \mathbb{R}, f \text{ continuous}\}.$$

Show that $(C_{\mathbb{R}}(X), \| \cdot \|_\infty)$ is a Banach space, but that neither $(C_{\mathbb{R}}(X), \| \cdot \|_1)$ nor $(C_{\mathbb{R}}(X), \| \cdot \|_2)$ is complete.

64. Let $E = \{f \in C_B(\mathbb{R}, \mathbb{R}) | f \text{ uniformly continuous}\}$. Show that E is a complete sub-algebra of $C_B(\mathbb{R}, \mathbb{R})$.

65. Let (E, d) be a complete metric space and $f : E \to E$. If there exists a positive integer r and $k \in (0, 1)$ such that

$$f^r = \underbrace{f \circ f \circ \cdots \circ f}_{r \text{ times}}$$

and $d(f^r(x), f^r(y)) \leq kd(x, y)$ for all $x, y \in E$, show that f has a unique fixed point.

66. Let $X = (0, \infty)$. Consider the function $\tilde{d} : X \times X \to \mathbb{R}_0^+$ defined by

$$\tilde{d}(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|.$$

   (a) Show that $\tilde{d}$ is a metric on X.
   (b) Show that $\tilde{d}$ and d_2 induce the same topology on X ($i.e.$ the open sets of X are exactly the same under both metrics).
   (c) Show that $(X, \tilde{d})$ is not a complete metric space.
   (d) Show that $((0, 1], \tilde{d})$ is a complete metric space.

67. Let $\mathcal{B}(X, \mathbb{R})$ denote the set of bounded functions from X to $\mathbb{R}$. It is easy to see that $\mathcal{B}(X, \mathbb{R})$ is a vector space over $\mathbb{R}$. The norm of $f \in \mathcal{B}(X, \mathbb{R})$ is defined by

$$\|f\| = \sup_{x \in X} |f(x)|.$$

   Show that $\mathcal{B}(X, \mathbb{R})$ is a Banach space with this norm.

68. Let (E, d) and $(F, \hat{d})$ be two metric spaces, and let $A \subseteq E$ be dense in E.
    (a) If $f : (A, d) \to (F, \hat{d})$ is continuous and if $\lim_{\mathbf{y}\to\mathbf{x},\mathbf{y}\in A} f(\mathbf{y})$ exists for all $\mathbf{x} \in E \setminus A$, show that there exists a unique continuous function $g : E \to F$ with $g|_A = f$.
    (b) Assume further that $(F, \hat{d})$ is complete. If $f : (A, d) \to (F, \hat{d})$ is uniformly continuous, show that there exists a unique function $g : E \to F$, uniformly continuous, with $g|_A = f$.
69. Let (E, d) be a metric space. Let $\mathcal{C}$ denote the set of Cauchy sequences in E.
    (a) i. Let $U = (\mathbf{u}_n), V = (\mathbf{v}_n) \in \mathcal{C}$. Show that $(d(\mathbf{u}_n, \mathbf{v}_n))$ converges, and denote its limit by $\delta(U, V)$.
        ii. Show that δ is symmetric and satisfies the triangle inequality.
    (b) Consider the equivalence relation $\sim$ on $\mathcal{C}$ defined by

$$U \sim V \Leftrightarrow \delta(U, V) = 0.$$

Write $\hat{E} = \mathcal{C}/\sim$ and denote the equivalence class of $U \in \mathcal{C}$ in $\hat{E}$ by $\hat{U}$.

i. What is the equivalence class of a sequence which converges in E?

ii. If $U \sim U'$ and $V \sim V'$, show that $\delta(U,V) = \delta(U',V')$. Thus, for $\hat{U}, \hat{V} \in \hat{E}$, the real number $\delta(\hat{U},\hat{V}) = \delta(U,V)$ is well-defined, not being dependent on the choice of class representatives.

iii. Show that δ is a metric on $\hat{E}$.

iv. Let $\iota : E \to \hat{E}$ be defined by $\iota(\alpha) = \widehat{(\alpha)}$, where (α) is the constant sequence. Show that ι is an isometry (and so also $1-1$). Furthermore, show that $\iota(E)$ is dense in $\hat{E}$.

(c) Show that $(\hat{E},\delta)$ is complete.

(d) Let (E_1,d_1) and (E_2,d_2) be complete metric spaces, and suppose that there are isometries $\iota_k : E \to E_k$ with $\iota_k(E)$ dense in E_k, for $k=1,2$. Show that there is a unique bijective isometry $\varphi : E_1 \to E_2$ such that $\varphi(\iota_1(\mathbf{x})) = \iota_2(\mathbf{x})$ for all $\mathbf{x} \in E$.

70. Let $A, B \subseteq E$, where E is endowed with any metric you care to imagine. Show that

(a) $A \subseteq \overline{A}$

(b) $\overline{(\overline{A})} = \overline{A}$

(c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(d) $\overline{\varnothing} = \varnothing$
(e) in general, $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$

71. Let A be a subset of (E, d). Show that $\overline{A} = \text{int}(A) \cup \partial A$.
72. Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}^\times\}$. Under the usual topology on $\mathbb{R}$, show that every point of A is a boundary point and that the only cluster point of A is 0.
73. Let

$$\tau_1 = \{U \subseteq \mathbb{R} \mid \mathbb{R} \setminus U \text{ is finite or } U = \varnothing\}$$

$$\tau_2 = \{U \subseteq \mathbb{R} \mid \mathbb{R} \setminus U \text{ is countable or } U = \varnothing\}$$

(a) Show that τ_1 and τ_2 define topologies on $\mathbb{R}$ (the **co-finite topology** and **countable complement** topology, respectively).
(b) What is the boundary of the set $A = \{\frac{1}{n} \mid n \in \mathbb{N}^\times\}$ under these two topologies?

74. Are the co-finite topologies and the countable complement topologies derived from a metric?
75. Let $A, B \subseteq (E, d)$. If $\mathbf{x} \in E$ is a cluster point of $A \cap B$, show that $\mathbf{x}$ is a cluster point of both A and B.

76. Let $\{H_n \mid n \in \mathbb{N}^\times\}$ be a family of closed subsets of a metric space (E, d) such that $\text{int}(H_n) = \varnothing$ for all $n \in \mathbb{N}^\times$. Assume further that E is such that $\text{int}(D(\mathbf{x}, \varepsilon)) \neq \varnothing$ for all $\mathbf{x} \in E$ and $\varepsilon > 0$. Let $G \subseteq_O E$.
   (a) If $\mathbf{x}_1 \in G \setminus H_1$, show that $\exists r_1 > 0$ such that $D(\mathbf{x}_1, r_1) \subseteq G$ and
   $$D(\mathbf{x}_1, r_1) \cap H_1 = \varnothing.$$
   (b) If $\mathbf{x}_2 \in \text{int}(D(\mathbf{x}_1, r_1)) \setminus H_2$, show that $\exists r_2 > 0$ such that
   $$D(\mathbf{x}_2, r_2) \subseteq \text{int}(D(\mathbf{x}_1, r_1)) \quad \text{and} D(\mathbf{x}_2, r_2) \cap H_2 = \varnothing.$$
   (c) Continue this process to obtain a nested family of closed subsets
   $$D(\mathbf{x}_1, r_1) \supsetneq D(\mathbf{x}_2, r_2) \supsetneq \cdots D(\mathbf{x}_n, r_n) \supsetneq D(\mathbf{x}_{n+1}, r_{n+1}) \supsetneq \cdots$$
   such that $H_n \cap D(\mathbf{x}_n, r_n) = \varnothing$ for all $n \in \mathbb{N}$. By the Cantor Intersection Theorem, $\exists \mathbf{x}_0 \in \bigcap D(\mathbf{x}_n, r_n)$. Conclude that G cannot be contained in $\bigcup H_n$.

This is a special case of the **Baire Category Theorem**.

77. A **line** in $\mathbb{R}^2$ is a set of points (x, y) which satisfy the equation $ax + by + c = 0$, where $(a, b) \neq \mathbf{0}$. Use the Baire Category Theorem to show that $\mathbb{R}^2$ is not a countable collection of lines.
78. Show that $B \subseteq (\mathbb{R}^p, d_2)$ is closed if and only if every convergent sequence in B converges to a point in B.
79. Let $(\mathbf{x}_n) \subseteq (\mathbb{R}^p, \|\cdot\|)$ such that

$$\|\mathbf{x}_{n+1} - \mathbf{x}_n\| \leq r\|\mathbf{x}_n - \mathbf{x}_{n-1}\|$$

where $r < 1$. Show that $(\mathbf{x}_n)$ converges.

Solutions

25. **Proof.**

(a)

(b)

(c)

(d)

(e)

(f) ∎

26. **Solution.**

(a)

(b)

27. **Proof.**

(a)

(b)

(c)

(d)

28. **Solution.**

29. **Solution.**

30. **Solution.**

31. **Proof.**

34. **Proof.**

(a)

(b)

(c)

(d)

51. **Proof.**

52. **Proof.**

53. **Proof.**

54. **Proof.**

55. **Proof.**

56.

57.

58. **Proof.**

59. **Proof.**

(a)

(b)

(c)

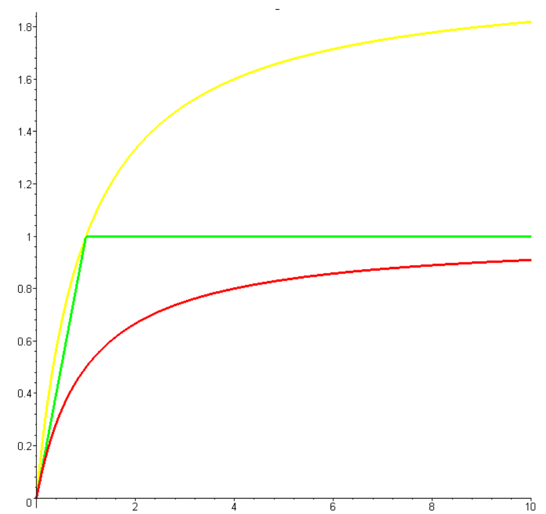
(d)

61. **Proof.**

(a)

(b)

(c)



68. **Proof.**

(a)

(b)

69. **Proof.**

(a) i.

ii.

(b) i.

ii.

iii.

iv.

(c)

(d)

70. **Proof.**

(a)

(b)

(c)

(d)

(e)

71. **Proof.**

72. **Proof.**

73. **Proof.**

(a)

(b)

75. **Proof.**

76. **Proof.**

(a)

(b)

(c)

77. **Proof.**

78. **Proof.**

79. **Proof.**